

A note about some general King-type operators

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ABSTRACT. In this note we give some conditions such that the general King operators defined in [3] provide a better approximation in a certain sense than the classical Bernstein operator for a class of decreasing and generalized convex functions. We present two particular cases and we show that these operators have some shape preserving properties.

KEY WORDS: King operator, Bernstein operator, shape preserving, lower approximation error, generalized convex functions

MSC 2000: 41A25, 41A36

1 Introduction

Let $C[0,1]$ be the linear space of all continuous real-valued function. The Bernstein operator $B_n : C[0, 1] \rightarrow C[0, 1]$ is given by

$$(B_n f)(x) = \sum_{k=0}^n p_{n,k}(x) f\left(\frac{k}{n}\right), \quad f \in C[0, 1], \quad x \in [0, 1],$$

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where

$$p_{n,k}(x) = \binom{n}{k} x^k (1-x)^{n-k}.$$

It is known that if $f \in C[0, 1]$ then the Bernstein operator converges uniformly to the function f , interpolates the function at the endpoints 0 and 1, has the shape preserving properties (if f is increasing or decreasing then $B_n f$ is the same and if f is convex or concave then $B_n f$ is the same). Also, we have

$$(1.1) \quad B_n(e_0)(x) = 1, \quad B_n(e_1)(x) = x, \quad B_n(e_2)(x) = x^2 + \frac{x(1-x)}{n}.$$

Thus, the Bernstein operator reproduces the functions e_0 and e_1 where $e_i(x) = x^i$, $i = 0, 1$.

In [5], King considered the following Bernstein type operator $V_n(\cdot, r_n) : C[0, 1] \rightarrow C[0, 1]$:

$$V_n(f; r_n)(x) = \sum_{k=0}^n p_{n,k}(r_n(x)) f\left(\frac{k}{n}\right), \quad f \in C[0, 1], \quad x \in [0, 1],$$

where $r_n : [0, 1] \rightarrow [0, 1]$ are continuous functions. For $r_n(x) = x$ one obtains the classical Bernstein operator. Using the properties of the Bernstein operator given in (1.1) we get

$$(1.2) \quad V_n(e_0; r_n)(x) = 1, \quad V_n(e_1; r_n)(x) = r_n(x),$$

$$(1.3) \quad V_n(e_2; r_n)(x) = r_n^2(x) + \frac{r_n(x)(1-r_n(x))}{n}.$$

King has shown that the operator $V_n(\cdot; r_n^*)$, with

$$r_n^*(x) = \begin{cases} x^2, & n = 1 \\ \frac{-1 + \sqrt{1 + 4n(n-1)x^2}}{2(n-1)}, & n > 1 \end{cases}$$

preserves the test functions e_0 and e_2 .

A generalization of the King operator $V_n(\cdot; r_n^*)$ was given in [2]. If

$$(1.4) \quad r_{n,\alpha}(x) = \frac{-1 - \alpha n + \sqrt{(\alpha n + 1)^2 + 4n(n-1)(x^2 + \alpha x)}}{2(n-1)},$$

with $\alpha \in [0, \infty)$ and $n \in \mathbb{N}$, $n > 1$, then the operator $V_n(\cdot; r_{n,\alpha})$ fixes the functions e_0 and $e_2 + \alpha e_1$. If $\alpha = 0$ then $V_n(\cdot; r_{n,\alpha})$ turn out to be King operator (i.e. $r_{n,0} = r_n^*$). In [2] it is shown that, if f belongs to a class of convex and increasing functions, then we have

$$(1.5) \quad f(x) \leq V_n(f; r_{n,\alpha})(x) \leq (B_n f)(x), \quad x \in [0, 1].$$

In [3] a general class of King type operators $V_n(\cdot; \tau_n)$ was constructed, where

$$\tau_n = (B_n \tau)^{-1} \circ \tau$$

and $\tau : [0, 1] \rightarrow [0, 1]$ is a continuous and strictly increasing function satisfying the conditions $\tau(0) = 0$ and $\tau(1) = 1$. These operators preserve the functions e_0 and τ . By taking $\tau = e_1$, $\tau = e_2$, $\tau = (e_2 + \alpha e_1)/(1 + \alpha)$ with $\alpha \geq 0$ we get the classical Bernstein operator and the operators from [5] and [2] respectively. If $\tau(x) = e^x$ then τ_n becomes the function

$$(1.6) \quad \varphi_n(x) = \frac{e^{x/n} - 1}{e^{1/n} - 1}$$

The corresponding operator $V_n(\cdot; \varphi_n)$ was considered in [1].

In this note we give some conditions such that the operator $V_n(\cdot; \tau_n)$ satisfies similar inequalities like in (1.5) for the class of functions which are decreasing and convex with respect to the function τ . This means that the operator $V_n(\cdot; \tau_n)$ provides lower approximation error than the classical Bernstein operator for the mentioned class of functions. We present two particular cases of this operator for which we give some results about the shape preserving property and the convergence.

2 Preliminaries

We list some results about the operator V_n which will be used in the next section.

Theorem 2.1 [3] *The sequence $\{\tau_n\}$ is uniformly convergent to e_1 , i.e.*

$$\lim_{n \rightarrow \infty} \|\tau_n - e_1\| = 0.$$

Theorem 2.2 [3] *For each $f \in C[0, 1]$ one has*

$$\lim_{n \rightarrow \infty} V_n(f; \tau_n) = f, \text{ uniformly on } [0, 1].$$

Taking into account the results in [2], we have the following shape preserving properties of the operator $V_n(\cdot; r_n)$:

Theorem 2.3 *Let $f \in C[0, 1]$ and $n \in \mathbb{N}$. If r_n is increasing and f is decreasing, then $V_n(f; r_n)$, $n \geq 1$ is decreasing, and if in addition r_n is concave and f is convex, then $V_n(f; r_n)$, $n > 1$ is convex.*

A function $f \in C[0, 1]$ is said to be convex with respect to the function τ if

$$\begin{vmatrix} 1 & 1 & 1 \\ \tau(x_0) & \tau(x_1) & \tau(x_2) \\ f(x_0) & f(x_1) & f(x_2) \end{vmatrix} \geq 0, \quad 0 \leq x_0 < x_1 < x_2 \leq 1$$

or equivalent to $f \circ \tau^{-1}$ is convex (in classical sense). We note that the convexity in the classical sense is obtained for $\tau = e_1$. For more about generalized convexity see [4].

As the operator $V_n(\cdot, \tau_n)$ fixes the functions e_0 and τ , the next theorem is a direct consequence of the Theorem 2 from [7].

Theorem 2.4 *If $f \in C[0, 1]$ is convex with respect to the function τ then*

$$f(x) \leq V_n(f; \tau_n)(x), \quad x \in [0, 1].$$

We have the following result of Voronovskaya type:

Theorem 2.5 [3] *Let $\tau \in C^2[0, 1]$.*

1) *For $f \in C^2[0, 1]$ and $x \in [0, 1]$ with $\tau'(x) > 0$*

(2.1)

$$\lim_{n \rightarrow \infty} n(V_n(f; \tau_n)(x) - f(x)) = -\frac{x(1-x)\tau''(x)}{2\tau'(x)}f'(x) + \frac{x(1-x)}{2}f''(x).$$

2) *Moreover, if $\tau'(x) > 0$, $x \in [0, 1]$, then the convergence in (2.1) is uniform on $[0, 1]$.*

3 Main result

Let $\tau : [0, 1] \rightarrow [0, 1]$ be a continuous and strictly increasing function satisfying the conditions $\tau(0) = 0$ and $\tau(1) = 1$.

Theorem 3.1 *If τ is concave and $f \in C[0, 1]$ is decreasing and convex with respect to the function τ , then we have*

$$f(x) \leq V_n(f; \tau_n)(x) \leq (B_n f)(x), \quad x \in [0, 1].$$

Proof. The first inequality follows from the Theorem 2.4.

Let $x \in [0, 1]$. Since τ is concave, we have $\tau(x) \geq B_n \tau(x)$. Using the fact that $(B_n \tau)^{-1}$ is increasing we obtain $((B_n \tau)^{-1} \circ \tau)(x) \geq x$ or equivalent $\tau_n(x) \geq x$. As f is decreasing, using the monotonicity property of the classical Bernstein operator, we get

$$V_n(f; \tau_n)(x) = (B_n f)(\tau_n(x)) \leq (B_n f)(x),$$

which completes the proof. ■

Remark 3.1 *The function $f \in C^2[0, 1]$ is convex with respect to the function τ if and only if*

$$f''(x) \geq \frac{\tau''(x)}{\tau'(x)}f'(x), \quad x \in [0, 1].$$

Next we present two particular cases.

Case 1. The function τ is a polynomial of degree at most two, i.e.

$$\tau(x) = ax^2 + bx + c, \quad x \in [0, 1], \quad a, b, c \in \mathbb{R}.$$

From the conditions $\tau(0) = 0$ and $\tau(1) = 1$ we get $c = 0$ and $b = 1 - a$. Thus

$$\tau(x) = ax^2 + (1 - a)x.$$

Imposing that τ to be strictly increasing and concave, we get $\tau'(x) = 2ax + 1 - a \geq 0$ and $\tau''(x) = 2a \leq 0$. It follows $a \in [-1, 0]$.

If $a = 0$ we have $\tau(x) = x$ and we get the Bernstein operator.

Let $a \in [-1, 0)$ and $n \in \mathbb{N}$, $n > 1$. Using (1.2) and (1.3) we get for every $x \in [0, 1]$

$$\begin{aligned} B_n(\tau)(x) &= B_n(ae_2 + (1 - a)e_1)(x) \\ &= aB_n(e_2)(x) + (1 - a)B_n(e_1)(x) \\ &= a\left(x^2 + \frac{x(1 - x)}{n}\right) + (1 - a)x. \end{aligned}$$

Then, for every $y \in [0, 1]$, we have

$$(B_n(\tau))^{-1}(y) = \frac{-a - n + an + \sqrt{(a + n - an)^2 + 4an(n - 1)y}}{2a(n - 1)}$$

and for every $x \in [0, 1]$ we get

$$\tau_n(x) = \tau_{n,a}^1(x) = \frac{-a - n + an + \sqrt{(a + n - an)^2 + 4anx(n - 1)(1 - a + ax)}}{2a(n - 1)}.$$

The King type operator is given by

$$V_n(f; \tau_{n,a}^1)(x) = \sum_{k=0}^n p_{n,k}(\tau_{n,a}^1(x)) f\left(\frac{k}{n}\right),$$

for every $f \in C[0, 1]$ and every $x \in [0, 1]$.

Remark 3.2 If $a = 1/(1 + \alpha)$, $\alpha \in (-\infty, -2]$ then $\tau_{n,a}^1 = r_{n,\alpha}$, where $r_{n,\alpha}$ is

given in (1.4). We get the same operator as in [2], but for the other choices of the parameter α .

From Theorem 2.1 and Theorem 2.2 we have that the sequence $\{V_n(f; \tau_{n,a}^1)\}$, $a \in [-1, 0)$ converges uniformly to the function f when $n \rightarrow \infty$, for every $f \in C[0, 1]$.

Lemma 3.1 *If $n \in \mathbb{N}$, $n > 1$ and $a \in [-1, 0)$ then*

- 1) $\tau_{n,a}^1$ is increasing and concave on $[0, 1]$,
- 2) $\tau_{n,a}^1(x)$ is decreasing with respect to a for every $x \in [0, 1]$.

Proof. For every $x \in [0, 1]$ and $a \in [-1, 0)$ we have

$$(\tau_{n,a}^1)'(x) = \frac{n(1 - a + 2xa)}{u_{n,a}(x)},$$

$$(\tau_{n,a}^1)''(x) = \frac{2an(a^2 - na^2 + n)}{(u_{n,a}(x))^{3/2}},$$

$$\frac{\partial}{\partial a} \tau_{n,a}^1(x) = \frac{n(-a - n + an + 2ax - 2anx + u_{n,a}(x))}{2a^2(n-1)u_{n,a}(x)},$$

with

$$u_{n,a}(x) = \sqrt{(a + n - an)^2 + 4anx(n-1)(1 - a + ax)}.$$

It follows $(\tau_{n,a}^1)' \geq 0$, $(\tau_{n,a}^1)'' < 0$ and $\frac{\partial}{\partial a} \tau_{n,a}^1 \leq 0$ on $[0, 1]$ and the statements are true. ■

The operator $V_n(\cdot; \tau_{n,a}^1)$, $a \in [-1, 0)$ has the following shape preserving properties:

Theorem 3.2 *Let $f \in C[0, 1]$, $a \in [-1, 0)$ and $n \in \mathbb{N}$, $n > 1$. If f is decreasing, then $V_n(f; \tau_{n,a}^1)$ is decreasing, and if in addition f is convex, then $V_n(f; \tau_{n,a}^1)$ is convex.*

Proof. We get the conclusion immediately by using Theorem 2.3 and 1) of Lemma 3.3. ■

The operator $V_n(\cdot; \tau_{n,a}^1)$ depends of the parameter a . We show that the choice $a = -1$ among all $a \in [-1, 0)$ yields to smallest approximation error for the operator $V_n(\cdot; \tau_{n,a}^1)$.

Theorem 3.3 *Let $f \in C[0, 1]$ be decreasing and convex with respect to the function τ and let $n \in \mathbb{N}$, $n > 1$. For $a \in [-1, 0)$ we have*

$$f(x) \leq V_n(f; \tau_{n,-1}^1)(x) \leq V_n(f; \tau_{n,a}^1)(x) \leq B_n f(x), \quad x \in [0, 1].$$

Proof. Let $a \in [-1, 0)$. The first and the last inequality follow from Theorem 3.1. Then, taking into account 2) of Lemma 3.1 we have that $\tau_{n,-1}^1(x) \geq \tau_{n,a}^1(x)$, $x \in [0, 1]$. Since f is decreasing, by using the monotonicity property of the classical Bernstein operator it follows

$$V_n(f; \tau_{n,-1}^1)(x) = B_n(f)(\tau_{n,-1}^1(x)) \leq B_n(f)(\tau_{n,a}^1(x)) = V_n(f; \tau_{n,a}^1)(x),$$

for every $x \in [0, 1]$, which ends the proof. ■

Remark 3.3 *If $f \in C^2[0, 1]$ and*

$$f''(x) \geq \frac{2a}{1+a} f'(x), \quad x \in [0, 1],$$

then the function f is convex with respect to the function τ .

We give two classes of functions which satisfy the conditions of the Theorem 3.3. For these functions, the operator V_n provides a smaller approximation error than the operator B_n .

Example 3.1 *Let $\theta > 0$, $\alpha < 0$ and $f : [0, 1] \rightarrow \mathbb{R}$, $f(x) = (x + \theta)^\alpha$. If*

$$\alpha \leq \frac{2a(1+\theta)}{a+1} + 1,$$

then the function f is decreasing and convex with respect to the function τ .

Example 3.2 *Let $\beta < 0$ and $f : [0, 1] \rightarrow \mathbb{R}$, $f(x) = e^{\beta x}$. If*

$$\beta \leq \frac{2a}{a+1},$$

then the function f is decreasing and convex with respect to the function τ .

The next theorem gives the quantitative estimate of the operator $V_n(\cdot; \tau_{n,a}^1)$.

Theorem 3.4 *Let $a \in [-1, 0)$ and $n \in \mathbb{N}$, $n > 1$. If $f \in C[0, 1]$ then*

$$|V_n(f; \tau_{n,a}^1)(x) - f(x)| \leq \left(1 + \frac{(\frac{a-1}{a} - 2x)(\tau_{n,a}^1(x) - x)}{\delta^2}\right) \omega_1(f, \delta),$$

for every $x \in [0, 1]$ and every $\delta > 0$, where $\omega_1(f, \delta)$ is the classical modulus of smoothness of order one:

$$\omega_1(f, \delta) = \sup\{|f(x+h) - f(h)| : x, x+h \in [0, 1], 0 \leq h \leq \delta\}.$$

Proof. Let $f \in C[0, 1]$. Using a version of the result of Shisha and Mond from [6] we have

$$\begin{aligned} |V_n(f; \tau_{n,a}^1)(x) - f(x)| &\leq |f(x)| |V_n(e_0; \tau_{n,a}^1)(x) - e_0(x)| \\ &+ \left(V_n(e_0; \tau_{n,a}^1)(x) + \frac{V_n(e_2 - 2xe_1 + x^2e_0; \tau_{n,a}^1)(x)}{\delta^2}\right) \omega_1(f, \delta), \end{aligned}$$

for every $x \in [0, 1]$ and every $\delta > 0$. Taking into account that

$$\begin{aligned} &V_n(e_2 - 2xe_1 + x^2e_0; \tau_{n,a}^1)(x) \\ &= V_n\left(e_2 + \frac{1-a}{a}e_1 - e_1\left(2x + \frac{1-a}{a}\right) + x^2e_0; \tau_{n,a}^1\right)(x) \\ &= x^2 + \frac{1-a}{a}x - \tau_{n,a}^1(x)\left(2x + \frac{1-a}{a}\right) + x^2 \end{aligned}$$

we get the conclusion. ■

We have the following results of Voronovskaya type:

Theorem 3.5

1) If $f \in C^2[0, 1]$ and $a \in (-1, 0)$, then

$$\lim_{n \rightarrow \infty} n(V_n(f; \tau_{n,a}^1)(x) - f(x)) = -\frac{ax(1-x)}{2ax+1-a}f'(x) + \frac{x(1-x)}{2}f''(x),$$

uniformly on $[0, 1]$,

2) If $f \in C^2[0, 1]$, then

$$\lim_{n \rightarrow \infty} n(V_n(f; \tau_{n,-1}^1)(x) - f(x)) = \frac{x}{2}f'(x) + \frac{x(1-x)}{2}f''(x),$$

uniformly on $[0, 1]$.

Proof. The proof follows using the Theorem 2.5. ■

Case 2. We consider

$$\tau(x) = ae^{bx} + c, \quad x \in [0, 1], \quad a, b, c \in \mathbb{R}.$$

From the conditions $\tau(0) = 0$ and $\tau(1) = 1$ we get $c = -a$ and $e^b = (1+a)/a$. It follows $a = 1/(e^b - 1)$ and $c = -1/(e^b - 1)$. Thus,

$$\tau(x) = \frac{e^{bx} - 1}{e^b - 1}.$$

In order to get τ strictly increasing and concave, we impose

$$\tau'(x) = \frac{be^{bx}}{e^b - 1} \geq 0$$

and

$$\tau''(x) = \frac{b^2e^{bx}}{e^b - 1} \leq 0,$$

whence $b \in (-\infty, 0)$.

If $b \rightarrow 0$ we have $\tau(x) \rightarrow x$ and we get the Bernstein operator.

Let $b \in (-\infty, 0)$ and $n \in \mathbb{N}$, $n \geq 1$. For every $x \in [0, 1]$ we have

$$B_n(\tau)(x) = B_n\left(\frac{e^{bx} - 1}{e^b - 1}\right)(x).$$

Taking into account that

$$\begin{aligned} B_n(e^{bx})(x) &= \sum_{k=0}^n \binom{n}{k} x^k (1-x)^{n-k} e^{bk/n} \\ &= \sum_{k=0}^n \binom{n}{k} (xe^{b/n})^k (1-x)^{n-k} \\ &= (xe^{b/n} + 1 - x)^n \end{aligned}$$

we get

$$B_n(\tau)(x) = \frac{(xe^{b/n} + 1 - x)^n - 1}{e^b - 1}.$$

Then,

$$(B_n(\tau))^{-1}(y) = \frac{(y(e^b - 1) + 1)^{1/n} - 1}{e^{b/n} - 1}, \quad y \in [0, 1]$$

and

$$\tau_n(x) = \tau_{n,b}^2(x) = \frac{e^{xb/n} - 1}{e^{b/n} - 1}, \quad x \in [0, 1].$$

The King type operator is given by

$$V_n(f; \tau_{n,b}^2)(x) = \sum_{k=0}^n p_{n,k}(\tau_{n,b}^2(x)) f\left(\frac{k}{n}\right)$$

for every $f \in C[0, 1]$ and $x \in [0, 1]$.

Remark 3.4 If we take $b = 1$ then $\tau_{n,b}^2 = \varphi_n$, where the function φ_n is given by (1.6). Since $1 \notin (-\infty, 0)$, the operator $V_n(\cdot, \varphi_n)$ doesn't belong to the class of the operators studied in this case.

If $f \in C[0, 1]$ then the uniform convergence of the sequence $\{V_n(f; \tau_{n,b}^2)\}$, $b \in (-\infty, 0)$ to the function f follows from Theorem 2.1 and Theorem 2.2.

Lemma 3.2 *If $n \in \mathbb{N}$, $n \geq 1$ and $b \in (-\infty, 0)$, then*

- 1) $\tau_{n,b}^2$ *is increasing and concave on $[0, 1]$,*
- 2) $\tau_{n,b}^2(x)$ *is decreasing with respect to b , for every $x \in [0, 1]$.*

Proof. For every $x \in [0, 1]$ and $b \in (-\infty, 0)$ we have

$$(\tau_{n,b}^2)'(x) = \frac{be^{xb/n}}{n(e^{b/n} - 1)} > 0,$$

$$(\tau_{n,b}^2)''(x) = \frac{b^2e^{xb/n}}{n^2(e^{b/n} - 1)} < 0,$$

$$\frac{\partial}{\partial b}\tau_{n,b}^2(x) = \frac{e^{b/n} + e^{(x+1)b/n}(-1+x) - e^{xb/n}x}{n(e^{b/n} - 1)^2} \leq 0$$

and the conclusions hold. ■

Theorem 3.6 *Let $f \in C[0, 1]$, $b \in (-\infty, 0)$ and $n \in \mathbb{N}$. If f is decreasing, then $V_n(f; \tau_{n,b}^2)$, $n \geq 1$ is decreasing, and if in addition f is convex, then $V_n(f; \tau_{n,b}^2)$, $n > 1$ is convex.*

Proof. The conclusion follows by using Theorem 2.3 and 1) of Lemma 3.2. ■

Theorem 3.7 *Let $f \in C[0, 1]$ be decreasing and convex with respect to the function τ . For $n \in \mathbb{N}$, $n \geq 1$ and $b_1, b_2 \in (-\infty, 0)$ with $b_1 < b_2$ we have*

$$f(x) \leq V_n(f; \tau_{n,b_1}^2)(x) \leq V_n(f; \tau_{n,b_2}^2)(x) \leq B_n f(x), \quad x \in [0, 1].$$

Proof. Let $b_1, b_2 \in (-\infty, 0)$, $b_1 < b_2$. We get the first and the last inequality using Theorem 3.1. Then, taking into account 2) of Lemma 3.9 we have that $\tau_{n,b_1}^1(x) \geq \tau_{n,b_2}^2(x)$, $x \in [0, 1]$. Since f is decreasing, by using the monotonicity property of the classical Bernstein operator it follows

$$V_n(f; \tau_{n,b_1}^2)(x) = B_n(f)(\tau_{n,b_1}^2(x)) \leq B_n(f)(\tau_{n,b_2}^2(x)) = V_n(f; \tau_{n,b_2}^2)(x),$$

for every $x \in [0, 1]$ which completes the proof. ■

Remark 3.5 *The convexity of a function $f \in C^2[0, 1]$ with respect to the function τ is equivalent to the inequality*

$$f''(x) \geq bf'(x), \quad x \in [0, 1].$$

Example 3.3 *Let $\theta > 0$, $\alpha < 0$ and $f : [0, 1] \rightarrow \mathbb{R}$, $f(x) = (x + \theta)^\alpha$. If $\alpha \leq b(1 + \theta) + 1$, then the function f satisfies the conditions of the Theorem 3.7. Thus, the operator V_n provides a better approximation of the function f than the operator B_n .*

Example 3.4 *Let $\beta < 0$ and $f : [0, 1] \rightarrow \mathbb{R}$, $f(x) = e^{\beta x}$. If $\beta \leq b$ then the function f satisfies the conditions of the Theorem 3.7. It follows that we get a better approximation of the function f for the operator V_n than for the operator B_n .*

Theorem 3.8 *Let $b \in (-\infty, 0)$ and $n \in \mathbb{N}$, $n \geq 1$. If $f \in C[0, 1]$, then*

$$\begin{aligned} & |V_n(f; \tau_{n,b}^2)(x) - f(x)| \\ & \leq \left(1 + \frac{1}{\delta^2}(\tau_{n,b}^2(x) - x)^2 + \frac{\tau_{n,b}^2(x)(1 - \tau_{n,b}^2(x))}{n\delta^2} \right) \omega_1(f, \delta) \end{aligned}$$

for every $x \in [0, 1]$ and every $\delta > 0$.

Proof. Let $f \in C[0, 1]$. From [6] we have

$$\begin{aligned} & |V_n(f; \tau_{n,b}^2)(x) - f(x)| \leq |f(x)| |V_n(e_0; \tau_{n,b}^2)(x) - e_0(x)| \\ & + \left(V_n(e_0; \tau_{n,b}^2)(x) + \frac{V_n(e_2 - 2xe_1 + x^2e_0; \tau_{n,b}^2)(x)}{\delta^2} \right) \omega_1(f, \delta) \end{aligned}$$

for every $x \in [0, 1]$ and every $\delta > 0$. Using (1.2) and (1.3) we get the conclusion. ■

Theorem 3.9 *If $f \in C^2[0, 1]$ and $b \in (-\infty, 0)$, then*

$$\lim_{n \rightarrow \infty} n(V_n(f; \tau_{n,b}^2)(x) - f(x)) = -\frac{x(1-x)b}{2}f'(x) + \frac{x(1-x)}{2}f''(x),$$

uniformly on $[0, 1]$.

Proof. The result of Voronovskaya type for the operator $V_n(\cdot; \tau_{n,b}^2)$ follows from Theorem 2.5. ■

The operators presented in this paper have similar properties as the classical Bernstein operator: they interpolate the function at the endpoints 0 and 1, preserve some test functions, have shape preserving properties. They provide a smaller approximation error than the Bernstein operator for the class of functions which are decreasing and convex with respect to the reproduced function.

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