

Hochschild cohomology of fully group-graded algebras as Mackey functor

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ABSTRACT. We prove that Hochschild cohomology of a certain class of fully group-graded algebras is a Mackey functor. We use the machinery of transfer maps between the Hochschild cohomology of symmetric algebras.

KEY WORDS: Hochschild cohomology, group, graded algebra, Mackey functor

1 Introduction

Let k be a field, G be a finite group and let R_G be a fully G -graded k -algebra. By definition, R_G has a decomposition $R_G = \bigoplus_{g \in G} R_g$, where each R_g is a k -vector space, for any $g \in G$, such that for all $g, h \in G$ we

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have $R_g R_h = R_{gh}$. In this paper we work only with symmetric fully G -graded algebras R_G with the property that for any subgroup H of G we have that R_H is a parabolic subalgebra of R_G , for further details see [1, Definition 2.3, Definition 5.1]. We begin by giving an example of a group-graded algebra which lies in the above mentioned class of symmetric algebras and which includes the group algebra case.

Example 1.1 *Let R_G be a fully G -graded algebra such that R_1 is a full matrix ring. By [4, Theorem 4] we know that R_G is a symmetric k -algebra. Moreover by [6, Lemma 1.1] we obtain that R_G is a crossed product. Furthermore, these statements assure us that for any subgroup H of G , R_H is a parabolic subalgebra of R_G .*

We use the language of transfer maps between Hochschild cohomology algebras of symmetric algebras, defined originally in [3], to give a structure of Mackey functor for Hochschild cohomology of such fully G -graded algebras. Recall from [3] that if M is an A – B -bimodule, projective as a left A -module and as a right B -module (where A, B are two symmetric k -algebras) there is a graded k -linear map, called transfer map

$$t_M : \mathrm{HH}^*(B) \longrightarrow \mathrm{HH}^*(A).$$

These transfer maps are also analyzed by explicit definitions in [5].

Let H be a subgroup of G and let $g \in G$. We denote by $M := R_G$ the R_H – R_G -bimodule structure on R_G given by multiplication in R_G ; similarly $N := R_G$ as an R_G – R_H -bimodule. Also we consider $P := R_{[gH]}$ to be the $R_{[gH]}$ – R_H -bimodule given by multiplication in R_G . By Lemma 2.1, a) (which is proved in the next section) we can define

$$r_H^G = t_M, \quad t_H^G = t_N, \quad c_{g,H} = t_P,$$

hence we have the following graded k -linear maps

$$\begin{aligned} r_H^G &: \mathrm{HH}^*(R_G) \rightarrow \mathrm{HH}^*(R_H); \\ t_H^G &: \mathrm{HH}^*(R_H) \rightarrow \mathrm{HH}^*(R_G); \\ c_{g,H} &: \mathrm{HH}^*(R_H) \rightarrow \mathrm{HH}^*(R_{[gH]}); \end{aligned}$$

which can be viewed as: "restriction", "transfer" and "conjugation" maps.

In the main theorem of this short note we verify that the quadruple

$$(\mathrm{HH}^*(kH), r_H^G, t_H^G, c_{g,H})_{H \leq G, g \in G}$$

is a Mackey functor, see [7, §53].

Theorem 1.1 *Let $K \leq H \leq G$ and $g, h \in G$. The following statements hold.*

- i) $r_K^H \circ r_H^G = r_K^G, t_H^G \circ t_K^H = t_K^G$;
- ii) $r_H^H = t_H^H = \mathrm{id}_{\mathrm{HH}^*(R_H)}$;
- iii) $c_{gh,H} = c_{g,hH} \circ c_{h,H}$;
- iv) $c_{h,H} = \mathrm{id}_{\mathrm{HH}^*(R_H)}$ if $h \in H$;
- v) $c_{g,K} \circ r_K^H = r_{gK}^H \circ c_{g,H}$ and $c_{g,H} \circ t_K^H = t_{gK}^H \circ c_{g,K}$;
- vi) $r_K^G \circ t_H^G = \sum_{g \in [K \backslash G / H]} t_{K \cap gH}^K \circ r_{K \cap gH}^H \circ c_{g,H}$, where $[K \backslash G / H]$ is a system of representatives of double cosets KgH with $g \in G$.

2 The proof of Theorem 1.1

In order to prove the main result, we need the following lemma. Although the results from the next lemma are easily checkable, for completeness we give the entire proof. We are inspired by the methods used in [2, Lemma 2.1].

Lemma 2.1 *Let $K, H \leq G$ and $g, h \in G$. The following statements hold.*

- a) M, N and P are projective as left and as right modules.
- b) $R_K \otimes_{R_{[K \cap gH]}} R_{[gH]} \cong R_{[KgH]}$ as $R_K - R_H$ -bimodules.
- c) $R_{[g(hH)]} \otimes_{R_{[hH]}} R_{[hH]} \cong R_{[ghH]}$ as $R_{[ghH]} - R_H$ -bimodules.

Proof.

For the proof of a) we easily see that the first two bimodules are projective left, respectively right modules since R_H is a parabolic subalgebra of R_G , see [1, Definition 5.1, (Pa2), (Pa1')]. Since $R_{[gH]}$ is a direct summand of R_G as a right R_H -module, and R_G is a projective right R_H -module (we use again the argument that R_H is parabolic) we obtain that $R_{[gH]}$ is a projective right R_H -module. Similar arguments show that $R_{[gH]}$ is a projective left $R_{[gH]}$ -module.

We prove assertion c). The map

$$\Phi : R_{[g(hH)]} \otimes_{R_{[hH]}} R_{[hH]} \rightarrow R_{[ghH]}$$

defined by $\Phi(r_1 \otimes r_2) = r_1 r_2$ for any $r_1 \otimes r_2 \in R_{[g(hH)]} \otimes_{R_{[hH]}} R_{[hH]}$ is a well-defined $R_{[ghH]} - R_H$ -bimodule homomorphism. The inverse of this map is

$$\Psi : R_{[ghH]} \rightarrow R_{[g(hH)]} \otimes_{R_{[hH]}} R_{[hH]},$$

given by $\Psi(r) = \sum_{i \in I} a_i \otimes b_i r$ for any $r \in R_{[ghH]}$, where $1 = \sum_{i \in I} a_i b_i$, and all $a_i \in R_g$, $b_i \in R_{g^{-1}}$, since $R_1 = R_g \cdot R_{g^{-1}}$; here I is a finite set of indices. Consider another decomposition $1 = \sum_{j \in J} a_j b_j$, where $a_j \in R_g$, $b_j \in R_{g^{-1}}$ for any $j \in J$ and J is a finite set of indices. Then the equalities

$$\begin{aligned} \sum_{i \in I} a_i \otimes b_i r &= \sum_{i \in I} 1 a_i \otimes b_i r = \sum_{i \in I} \sum_{j \in J} a_j b_j a_i \otimes b_i r \\ &= \sum_{j \in J} a_j \otimes b_j \sum_{i \in I} a_i b_i r = \sum_{j \in J} a_j \otimes b_j r \end{aligned}$$

show that the last mentioned map is well-defined.

At last, for the assertion b) we fix $t \in K$ and $z \in H$ and we observe that

$$R_t \otimes_{R_1} R_{gz} \cong R_{tgz}$$

as $R_1 - R_1$ -bimodules. Indeed, this can similarly be shown as statement c), the map given by multiplication has an inverse that is defined using the relation $1 = \sum_{i \in I} a_i b_i$, where for all $i \in I$ we have $a_i \in R_t$ and $b_i \in R_{t^{-1}}$. One can observe that, by linearity, the multiplication map extends to a $R_K - R_H$ -bimodule isomorphism between $R_K \otimes_{R_{[K \cap gH]}} R_{[gH]}$ and $R_{[KgH]}$.

Proof. (of Theorem 1.1):

ii) and iv) are an easy exercise if we use [5, Proposition 2.7 (4)]. Statement v) is similar to iii) and is left for the reader. Also, the second part of (i) is analogous to the first part and is left as an exercise. The rest of the proof is a consequence of [5, Proposition 2.7 (1)] and of the bimodule isomorphisms from Lemma 2.1. Let $X := R_H$, viewed as an $R_K - R_H$ -bimodule, $Y := R_{[ghH]}$ viewed as an $R_{[ghH]} - R_H$ -bimodule, $Z := R_{[g(hH)]}$ viewed as an $R_{[ghH]} - R_{[hH]}$ -bimodule and $U := R_{[hH]}$ viewed as an $R_{[hH]} - R_H$ -bimodule. For (i) we have

$$r_K^H \circ r_H^G = t_X \circ t_M = t_{X \otimes_{R_H} M} = r_K^G.$$

For iii) we have

$$\begin{aligned} c_{gh,H} &= t_Y \text{ and} \\ c_{g,hH} \circ c_{h,H} &= t_Z \circ t_U = t_{Z \otimes_{R_{[hH]}} U}; \end{aligned}$$

From Lemma 2.1, c) we know that $Y \cong Z \otimes_{R_{[hH]}} U$ as $R_{[ghH]} - R_H$ -bimodules, hence iii). Consider the $R_K - R_G$ -bimodule $A := R_G$, the $R_K - R_H$ -bimodule $B := R_G$, the $R_K - R_{[K \cap^g H]}$ -bimodule $C := R_K$ and the $R_K - R_H$ -bimodule $D := R_{[K^g H]}$. For vi) we have

$$r_K^G \circ t_H^G = t_A \circ t_M = t_{A \otimes_{R_G} M} = t_B,$$

since $A \otimes_{R_G} M \cong B$.

By [3, Proposition 2.11 (i), 2.12 (iii)] and since $C \otimes_{R_{[K \cap^g H]}} P \cong D$ as $R_K - R_H$ -bimodules (see Lemma 2.1, b)) the following equalities hold

$$\begin{aligned} \sum_{g \in [K \setminus G/H]} t_{K \cap^g H}^K \circ r_{K \cap^g H}^{gH} \circ c_{g,H} &= \sum_{g \in [K \setminus G/H]} t_{C \otimes_{R_{[K \cap^g H]}} R_{[gH]} \otimes_{R[gH]} P} \\ &= \sum_{g \in [K \setminus G/H]} t_{C \otimes_{R_{[K \cap^g H]}} P} \\ &= \sum_{g \in [K \setminus G/H]} t_D \\ &= t_B. \end{aligned}$$

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