

A simple proof of a generalized Bernstein Voronovskaja-type Theorem

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ABSTRACT. In this short note, we present a simple, self-contained proof of a generalized Bernstein Voronovskaja-type theorem.

KEY WORDS: Bernstein-Voronovskaja Theorem; Positive linear operators; Rate of convergence.

MSC 2000: 41A10, 41A25, 41A36, 41A17, 41A80.

1 Introduction

Let $f: [0, 1] \rightarrow \mathbb{R}$. For any non-negative integer n , the classical Bernstein operator $B_n: C[0, 1] \rightarrow C[0, 1]$ is defined by

$$B_n(f; x) = \sum_{k=0}^n \binom{n}{k} x^k (1-x)^{n-k} f\left(\frac{k}{n}\right), \quad x \in [0, 1].$$

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In 1932, S. N. Bernstein gave the following generalization of the Voronovskaja [6] result.

Theorem 1.1 (Bernstein [1]) *If $q \in \mathbb{N}$ is even and $f \in C^q[0, 1]$, then,*

$$\lim_{n \rightarrow \infty} n^{q/2} \left(B_n(f; x) - \sum_{r=0}^q B_n((\cdot - x)^r; x) \frac{f^{(r)}(x)}{r!} \right) = 0,$$

uniformly on $[0, 1]$.

It is worthwhile mentioning that, in its complete form (i.e., for all positive integers q), Bernstein's Theorem 1.1 was proved only in 2012 ([5, 3]).

Let o denotes Landau's "little- o " notation. In [4, Corollary 8], we have proved that for any positive linear operator preserving constants $L: C[0, 1] \rightarrow C[0, 1]$, $q \in \mathbb{N}$, $f \in C^q[0, 1]$ and $x \in [0, 1]$, the following inequality holds true,

$$\begin{aligned} & \left| L(f; x) - \sum_{i=0}^q L((\cdot - x)^i; x) \frac{f^{(i)}(x)}{i!} \right| \\ & \leq \frac{\sqrt{L((\cdot - x)^{2q}; x)}}{q!} \tilde{\omega} \left(f^{(q)}; \frac{\sqrt{L((\cdot - x)^2; x)}}{q+1} \right), \end{aligned}$$

where $\tilde{\omega}(f^{(q)}; \delta)$, is the *least concave majorant of the modulus of continuity of $f^{(q)}$* . As a consequence, we have obtained the following theorem.

Theorem 1.2 ([4, Theorem 10]) *If $L_n: C[0, 1] \rightarrow C[0, 1]$ is a sequence of positive linear approximation operators, $f \in C^q[0, 1]$ and $x \in [0, 1]$, then*

$$L_n(f; x) - \sum_{i=0}^q L_n((\cdot - x)^i; x) \frac{f^{(i)}(x)}{i!} = o \left(\sqrt{L_n((\cdot - x)^{2q}; x)} \right), \quad \text{as } n \rightarrow \infty.$$

In this short note we give a simple self-contained proof of Theorem 1.2.

2 The proof

Suppose that $f \in C[0, 1]$ is n times differentiable at a point $x \in [0, 1]$. We use the Peano form of the remainder in Taylor's formula. Define the function $h_x \in C[0, 1]$ by

$$(2.1) \quad f - \sum_{i=0}^q (\cdot - x)^i \frac{f^{(i)}(x)}{i!} = (\cdot - x)^q h_x,$$

and $h_x(x) = 0$. By using the Cauchy-Schwarz inequality, from Eq. (2.1), we deduce

$$\begin{aligned} & \left| L_n(f; x) - \sum_{i=0}^q L_n((\cdot - x)^i; x) \frac{f^{(i)}(x)}{i!} \right| \\ &= \left| L_n((\cdot - x)^q h_x; x) \right| \leq \sqrt{L_n((\cdot - x)^{2q}; x)} \sqrt{L_n(h_x^2; x)}. \end{aligned}$$

Since $\lim_{n \rightarrow \infty} L_n(h_x^2; x) = h_x^2(x) = 0$, the proof is complete. $\square$

Remark 2.1 As mentioned in [4], since $B_n((\cdot - x)^{2q}, x) = \mathcal{O}(n^{-q})$, (see, e.g., [2, Theorem 1.1, p. 303]), Theorem 1.2 is a generalization of the complete form of Bernstein's Theorem 1.1.

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