

A generalized interior approach to constrained set-valued duality

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ABSTRACT. This article addresses a new approach to duality for constrained set-valued optimization problems, by means of solutions defined with the help of the nonempty quasi interior of a convex cone, employing a set-type criterion. Weak duality theorems and theorems containing optimality conditions are presented, along with an application.

KEY WORDS: set-valued duality, quasi-interior, optimality conditions

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1 Introduction

From a theoretical point of view, a general constrained set-valued optimization problem

$$(P_0^{sv}) \quad \min_{G(x) \cap -C \neq \emptyset} F(x),$$

can be approached either by using the so-called **vector criterion**, or by developing the so-called **set criterion**.

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Several authors tackled the set-valued optimization theory from perspectives involving the extension of results concerning vector-valued functions. The vector criterion employs the determination of vector-like efficient points ((Pareto)-efficient, weakly-efficient, strongly-efficient, etc.) within the entire set

$$\mathcal{F} = \bigcup \{F(x) : x \in X, G(x) \cap -C \neq \emptyset\}.$$

This means that $\bar{x} \in X$ such that $G(\bar{x}) \cap -C \neq \emptyset$, is an efficient solution to (P_0^{sv}) if there exists an $\bar{y} \in F(\bar{x})$ such that $\bar{y}$ is an efficient element to the set $\mathcal{F}$. Such an approach can be found in H. W. CORLEY [5], T. TANINO [14], T. TANINO and Y. SAWARAGI [15], W. SONG [12], [13]. In Chapter 7 of R. I. BOT, S. M. GRAD and G. WANKA [4] is presented a survey on set-valued duality using different vector-type extended notions.

The **set criterion** involves a direct comparison of the sets $F(x)$, for all $x \in X$, $G(x) \cap -C \neq \emptyset$ and it is based on an ordering relation on $\mathcal{P}(Y)$ rather than on an order on Y , case encountered when working with a vector criterion. This is the reason why, at times, the set criterion is also called the **natural criterion** in set-valued optimization. Connected to it, T. KUROIWA published a series of articles [8], [9], [10]. We also mention KUROIWA D., TANAKA T. and TRUONG X. D. H. [11]. KUROIWA's approach was expanded in several recent papers written by E. HERNÁNDEZ and L. RODRÍGUEZ-MARIN [7] and M. ALONSO and L. RODRÍGUEZ-MARIN [1].

In this article we propose a new duality perspective for constrained set-valued optimization problems, by employing efficient solutions defined with the help of the quasi interior of a convex cone, continuing the theory developed in [6] for unconstrained set-valued optimization problems.

Section 2, entitled Preliminaries, familiarizes the reader with the notions and results taken from the specialized literature. We present definitions and characterization properties for the quasi relative interior and quasi interior of a set. Two set-relations defined with the help of the quasi interior are presented along with some of their properties. A qi-conjugate function associated with a set-valued function, and a qi-subdifferential, in analogy to the conjugate functions and subdifferential from scalar optimization are mentioned as well.

Using a general perturbation approach, we construct in Section 3 a new set-valued duality theory for constrained problems, employing qi-efficient solutions. Our results are more general than those of E. HERNÁNDEZ and L. RODRÍGUEZ-MARIN [7] for weak efficiency, since the quasi interior of a set is a more general notion than the interior.

The article ends with a section devoted to an application of our set-valued Lagrange duality theorem by means of qi-efficiency, application stated in $\ell^2(\mathbb{R})$.

2 Preliminaries

Suppose that X is a topological vector space, and let X^* be the topological dual space of X . Given a linear continuous functional $x^* \in X^*$ and a point $x \in X$, we denote by $\langle x^*, x \rangle$ the value of x^* at x .

The **normal cone** associated with a set $M \subseteq X$ is defined by

$$N_M(x) := \begin{cases} \{x^* \in X^* : \langle x^*, y - x \rangle \leq 0 \text{ for all } y \in M\} & \text{if } x \in M \\ \emptyset & \text{otherwise.} \end{cases}$$

Given a nonempty cone $C \subseteq X$, its **dual cone** is the set

$$C^+ := \{x^* \in X^* : \langle x^*, x \rangle \geq 0 \text{ for all } x \in C\}.$$

Consider now a separated topological vector space X , and let $M \subseteq X$ be a set. The **quasi-relative interior** of M is

$$\text{qri}M := \{x \in M : \text{clcone}(M - x) \text{ is a linear subspace of } X\}$$

(see J. M. BORWEIN and A. S. LEWIS [2]).

The **quasi interior** of a set $M \subseteq X$ is tightly connected to the quasi-relative interior and appeared in the literature prior to it. It is defined by

$$\text{qi}M := \{x \in M : \text{clcone}(M - x) = X\}.$$

When M is a convex set, then

$$\text{qi}M \subseteq \text{qri}M \text{ and } \text{qri}\{x\} = \{x\} \text{ for all } x \in X.$$

Moreover, whenever $\text{qi}M \neq \emptyset$, then $\text{qi}M = \text{qri}M$.

When X is a separated locally convex space, and $M \subseteq X$ is a convex set, then the following chain of inclusions holds:

$$(2.1) \quad \text{int}M \subseteq \text{core}M \subseteq \text{qi}M \subseteq \text{qri}M \subseteq M.$$

When $\text{int}M \neq \emptyset$, then all the generalized interior notions in (2.1) collapse in equality to $\text{int}M$, i.e. the topological interior of M .

We continue by presenting some characterizations of the quasi interior and quasi-relative interior of convex sets in separated locally convex spaces.

Theorem 2.1 (J. M. BORWEIN, A. S. LEWIS) *Let M be a convex subset of a separated locally convex space X , and let $x \in M$. Then $x \in \text{qri}M$ if and only if $N_M(x)$ is a linear subspace of X^* .*

Theorem 2.2 (R. I. BOŤ, E. R. CSETNEK, G. WANKA [3]) *Let M be a convex subset of a separated locally convex space X , and let $x \in M$. Then $x \in \text{qi}M$ if and only if $N_M(x) = \{0\}$.*

For two convex subsets M and N of a separated locally convex X such that $M \subseteq N$, it holds

$$(2.2) \quad \text{qi}M \subseteq \text{qi}N,$$

a property which is no longer true for the quasi-relative interior.

Let X be a separated locally convex space, and let $C \subseteq X$ be a convex cone. Then, we deduce that the equality

$$(2.3) \quad \text{qi}C + C = \text{qi}C$$

holds.

Proposition 2.1 *Let C be a nonempty convex cone of a separated locally convex space X . Then, for all $x^* \in C^+ \setminus \{0\}$ and for all $x \in \text{qi}C$, the following inequality holds:*

$$(2.4) \quad \langle x^*, x \rangle > 0.$$

Proof. The case when $\text{qi}C = \emptyset$ is obvious.

We continue with the case when $\text{qi}C \neq \emptyset$, and use an approach similar to the proof given by R. I. BOT, S. M. GRAD and G. WANKA. [4, Proposition 2.1.1]. By contradiction we assume that there exist an $x_0^* \in C^+ \setminus \{0\}$ and an $x_0 \in \text{qi}C$ such that

$$(2.5) \quad \langle x_0^*, x_0 \rangle = 0.$$

From the definition of the dual cone C^+ we know that

$$(2.6) \quad \langle x_0^*, x \rangle \geq 0 \text{ for all } x \in C.$$

Combining (2.5) and (2.6) we get

$$\langle -x_0^*, x - x_0 \rangle \leq 0 \text{ for all } x \in C,$$

which means that $-x_0^* \in N_C(x_0)$. As $x_0 \in \text{qi}C$, according to Theorem 2.2, we have $N_C(x_0) = \{0\}$. This implies that $-x_0^* = 0$, which is a contradiction. Thus (2.4) holds. ■

2.1 Two Set Relations Defined by Means of the Quasi Interior

We consider the following framework:

$$(2.7) \quad \begin{cases} Y \text{ is a separated locally convex space;} \\ K \subset Y \text{ is a pointed, convex cone with } \text{qi}K \neq \emptyset. \end{cases}$$

Let us recall that

$$\mathcal{P}_0(Y) := \{A : A \subseteq Y \text{ and } A \neq \emptyset\}.$$

We start by presenting some set relations defined with the help of a convex cone, introduced by D. KUROIWA.

Definition 2.1 (D. KUROIWA) *Let the hypotheses (2.7) be satisfied, and let A and B belong to $\mathcal{P}_0(Y)$. Then we write:*

$$(a) \quad A \leq^l B \text{ if } B \subseteq A + K;$$

- (b) $A \leq^u B$ if $A \subseteq B - K$;
- (c) $A \sim^l B$ if $A \leq^l B$ and $B \leq^l A$.

D. KUROIWA [8] proved that $\sim^l$ is an equivalence relation on $\mathcal{P}_0(Y)$. ■

Definition 2.2 (A. GRAD [6]) *Let the framework (2.7) be satisfied, and let A and B belong to $\mathcal{P}_0(Y)$. Then we write:*

- (a) $A \leq_{\text{qi}K}^l B$ if $B \subseteq A + \text{qi}K$;
- (b) $A \leq_{\text{qi}K}^u B$ if $A \subseteq B - \text{qi}K$.

The definitions above justify the choice made in (2.7) for $K \neq Y$. If $K = Y$ the new two set relations $\leq_{\text{qi}K}^l$ and $\leq_{\text{qi}K}^u$ would be useless.

The relations $\leq_{\text{qi}K}^l$ and $\leq_{\text{qi}K}^u$ are transitive.

With the help of the relations introduced in Definition 2.2 we defined four new efficiency notions for sets.

Definition 2.3 (A. GRAD [6]) *Let the hypotheses (2.7) be satisfied, and let $\mathcal{S} \subseteq \mathcal{P}_0(Y)$. A set $A \in \mathcal{S}$ is said to be:*

- (a) an $l - \text{Min}_{\text{qi}}$ -**efficient set** of $\mathcal{S}$, if for each set $B \in \mathcal{S}$ satisfying

$$B \leq_{\text{qi}K}^l A, \text{ the relation } A \leq_{\text{qi}K}^l B \text{ holds.}$$

- (b) an $l - \text{Max}_{\text{qi}}$ -**efficient set** of $\mathcal{S}$, if for each set $B \in \mathcal{S}$ satisfying

$$A \leq_{\text{qi}K}^l B, \text{ the relation } B \leq_{\text{qi}K}^l A \text{ holds.}$$

- (c) an $u - \text{Min}_{\text{qi}}$ -**efficient set** of $\mathcal{S}$, if for each set $B \in \mathcal{S}$ satisfying

$$B \leq_{\text{qi}K}^u A, \text{ the relation } A \leq_{\text{qi}K}^u B \text{ holds.}$$

- (d) an $u - \text{Max}_{\text{qi}}$ -**efficient set** of $\mathcal{S}$, if for each set $B \in \mathcal{S}$ satisfying

$$A \leq_{\text{qi}K}^u B, \text{ the relation } B \leq_{\text{qi}K}^u A \text{ holds.}$$

The sets of all $l - \text{Min}_{\text{qi}}$ -efficient, $l - \text{Max}_{\text{qi}}$ -efficient, $u - \text{Min}_{\text{qi}}$ -efficient and $u - \text{Max}_{\text{qi}}$ -efficient sets of $\mathcal{S}$ are denoted by

$$l - \text{Min}_{\text{qi}}\mathcal{S}, l - \text{Max}_{\text{qi}}\mathcal{S}, u - \text{Min}_{\text{qi}}\mathcal{S} \text{ and } u - \text{Max}_{\text{qi}}\mathcal{S},$$

respectively.

Remark 2.1 E. HERNÁNDEZ and L. RODRÍGUEZ-MARIN [7] considered some weak set-efficiency notions defined with $\leq^l$ when dealing with convex cones with nonempty topological interior. Our notions are more general and can be used even in the case when the topological interior of the cones is empty. ■

Let the hypotheses (2.7) be satisfied, and let $\mathcal{S} \subseteq \mathcal{P}_0(Y)$. From Definition 2.3 it follows that

$$y + l - \text{Min}_{\text{qi}}\mathcal{S} = l - \text{Min}_{\text{qi}}(y + \mathcal{S}) \text{ for all } y \in Y.$$

Similar equalities hold for $l - \text{Max}_{\text{qi}}\mathcal{S}$, $u - \text{Min}_{\text{qi}}\mathcal{S}$ and $u - \text{Max}_{\text{qi}}\mathcal{S}$, respectively. ■

Proposition 2.2 Let the hypotheses (2.7) be satisfied, and let $\mathcal{S} \subseteq \mathcal{P}_0(Y)$. Then the following equality holds:

$$(2.8) \quad l - \text{Min}_{\text{qi}}(-\mathcal{S}) = -\mathcal{S},$$

where $-\mathcal{S} = \{-A : A \in \mathcal{S}\}$.

Proof. “ $\subseteq$ ” Let $A \in \mathcal{S}$ be such that $-A \in l - \text{Min}_{\text{qi}}(-\mathcal{S})$. We prove that $A \in \mathcal{S}$. To this end, consider $B \in \mathcal{S}$ such that $A \leq_{\text{qiK}}^u B$. This is equivalent to $-B \leq_{\text{qiK}}^l -A$. Then, from $-A \in l - \text{Min}_{\text{qi}}(-\mathcal{S})$ it follows that $-A \leq_{\text{qiK}}^l -B$, which in its turn, is equivalent to $B \leq_{\text{qiK}}^u A$. Hence $-A \in -u - \text{Max}_{\text{qi}}\mathcal{S}$. Therefore

$$l - \text{Min}_{\text{qi}}(-\mathcal{S}) \subseteq -u - \text{Max}_{\text{qi}}\mathcal{S}.$$

“ $\supseteq$ ” The inclusion $-u - \text{Max}_{\text{qi}}\mathcal{S} \subseteq l - \text{Min}_{\text{qi}}(-\mathcal{S})$ can be proved similarly. ■

2.2 qi-Conjugate Functions and qi-Subgradients

We consider the following hypotheses:

$$(2.9) \quad \begin{cases} X \text{ is a topological vector space;} \\ Y \text{ is a separated locally convex space;} \\ K \subset Y \text{ is a pointed, convex cone with } \text{qi}K \neq \emptyset; \\ F : X \rightarrow \mathcal{P}(Y) \text{ is a proper set-valued function.} \end{cases}$$

Let us recall that under the hypotheses (2.9) the **domain** of the set-valued function F is the set

$$\text{dom}F := \{x \in X : F(x) \neq \emptyset\}.$$

The function F is said to be **proper** if $\text{dom}F \neq \emptyset$.

Definition 2.4 (A. GRAD [6]) *Let the hypotheses (2.9) be satisfied. The **qi-conjugate** function of F is the set-valued function $F_{\text{qi}K}^* : \mathcal{L}(X, Y) \rightarrow \mathcal{P}(\mathcal{P}(Y))$ defined by*

$$F_{\text{qi}K}^*(T) := \text{u} - \text{Max}_{\text{qi}}\{Tx - F(x) : x \in X\} \text{ for all } T \in \mathcal{L}(X, Y).$$

Taking into consideration Definition 2.3 (d) and the hypotheses (2.9), it results that the following equality holds:

$$F_{\text{qi}K}^*(T) = \text{u} - \text{Max}_{\text{qi}}\{Tx - F(x) : x \in \text{dom}F\}.$$

In the following we prove a result which may be regarded as an **extension of the Fenchel-Young inequality** to this set-valued setting.

Theorem 2.3 (A. GRAD [6]) *Let the hypotheses (2.9) be satisfied, let $x_0, x_1 \in \text{dom}F$, and let $T \in \mathcal{L}(X, Y)$ be such that*

$$(2.10) \quad F(x_1) - Tx_1 \in -F_{\text{qi}K}^*(T).$$

Then the following statements hold:

- (a) *If $F(x_0) - Tx_0 \leq_{\text{qi}K}^l F(x_1) - Tx_1$, then $F(x_1) - Tx_1 \leq_{\text{qi}K}^l F(x_0) - Tx_0$.*

(b) If $F(x_0) - Tx_0 \leq_{\text{qiK}}^l F(x_1) - Tx_1$, then $F(x_1) - Tx_1 \sim^l F(x_0) - Tx_0$.

With the help of the qi-conjugate function we extend the notions of the subgradient and subdifferential to set-valued functions.

Definition 2.5 (A. GRAD [6]) *Let the hypotheses (2.9) be satisfied, and let $\bar{x} \in \text{dom}F$.*

(a) *An operator $T \in \mathcal{L}(X, Y)$ is said to be a **qi-subgradient** of the set-valued function F at $\bar{x}$ if*

$$T\bar{x} - F(\bar{x}) \in F_{\text{qiK}}^*(T).$$

(b) *The set of all qi-subgradients of the set-valued function F at $\bar{x}$ is called the **qi-subdifferential** of F at $\bar{x}$ and is denoted by $\partial_{\text{qiK}}F(\bar{x})$.*

By convention, if $\bar{x} \notin \text{dom}F$, then we consider that $\partial_{\text{qiK}}F(\bar{x}) = \emptyset$.

In light of Definition 2.5, the condition (2.10) in Theorem 2.3 can be equivalently rewritten as $T \in \partial_{\text{qiK}}F(x_1)$.

Similar to the scalar and vector case, we prove the following property.

Proposition 2.3 (A. GRAD [6]) *Let the hypotheses (2.9) be satisfied, and let $\bar{x} \in \text{dom}F$. Then*

$$F(\bar{x}) \in 1 - \text{Min}_{\text{qi}}\{F(x) : x \in X\} \text{ if and only if } 0 \in \partial_{\text{qiK}}F(\bar{x}).$$

3 A Quasi Interior Perturbation Approach in Constrained Set-Valued Optimization

We consider the general set-valued optimization problem with cone constraints

$$(CP_{\text{qi}}^{sv}) \quad \begin{array}{l} 1 - \text{Min}_{\text{qi}} F(x) \\ G(x) \cap -C \neq \emptyset \end{array}$$

stated under the following hypotheses:

$$(3.1) \quad \begin{cases} X \text{ and } W \text{ are topological vector spaces;} \\ Y \text{ and } Z \text{ are separated locally convex spaces;} \\ K \subset Y \text{ is a pointed, convex cone with } \text{qi}K \neq \emptyset; \\ C \subset Z \text{ is a nonempty, pointed and convex cone;} \\ F : X \rightarrow \mathcal{P}(Y) \text{ and } G : X \rightarrow \mathcal{P}(Z) \text{ are proper set-valued functions;} \\ \{x \in (\text{dom}F) \cap (\text{dom}G) : G(x) \cap (-C) \neq \emptyset\} \neq \emptyset. \end{cases}$$

In the following we use the notation

$$\mathcal{A}_{CP_{\text{qi}}^{sv}} := \{x \in (\text{dom}F) \cap (\text{dom}G) : G(x) \cap (-C) \neq \emptyset\}.$$

A **perturbation function** associated with F is actually a set-valued function $\Phi_C^{sv} : X \times W \rightarrow \mathcal{P}(Y)$ such that

$$\Phi_C^{sv}(x, 0) = \begin{cases} F(x) & \text{if } x \in \mathcal{A}_{CP_{\text{qi}}^{sv}} \\ \emptyset & \text{otherwise.} \end{cases}$$

A set-valued dual problem associated with (CP_{qi}^{sv}) with the help of Φ_C^{sv} is

$$(CD_{\text{qi}}^{sv}) \quad 1 - \text{Max}_{\text{qi}} \left[-(\Phi_C^{sv})_{\text{qi}K}^*(0, T) \right].$$

$T \in \mathcal{L}(W, Y)$

In the following we use the notation

$$\mathcal{A}_{CD_{\text{qi}}^{sv}} := \{(T, x, w) \in \mathcal{L}(W, Y) \times X \times W : (0, T) \in \partial_{\text{qi}K} \Phi_C^{sv}(x, w)\}.$$

Definition 3.1 *Let the hypotheses (3.1) be satisfied. An operator $\tilde{T} \in \mathcal{L}(W, Y)$ is said to be a **qi-efficient solution** to the dual problem (CD_{qi}^{sv}) if there exists an $(\tilde{x}, \tilde{w}) \in \text{dom} \Phi_C^{sv}$ such that*

$$(3.2) \quad (\tilde{T}, \tilde{x}, \tilde{w}) \in \mathcal{A}_{CD_{\text{qi}}^{sv}}$$

and

$$(3.3) \quad -\tilde{T}\tilde{w} + \Phi_C^{sv}(\tilde{x}, \tilde{w}) \in 1 - \text{Max}_{\text{qi}} \{-Tw + \Phi_C^{sv}(x, w) : (T, x, w) \in \mathcal{A}_{CD_{\text{qi}}^{sv}}\}.$$

With the help of the following **set-valued weak duality** theorem we

certify that (CD_{qi}^{sv}) is actually a dual problem to (CP_{qi}^{sv}) .

Theorem 3.1 *Let the hypotheses (3.1) be satisfied, let $x_0 \in \mathcal{A}_{CP_{\text{qi}}^{sv}}$, and let $(T, x, w) \in \mathcal{A}_{CD_{\text{qi}}^{sv}}$. Then the following statements are true:*

- (a) *If $F(x_0) \leq_{\text{qiK}}^l -Tw + \Phi_C^{sv}(x, w)$, then $-Tw + \Phi_C^{sv}(x, w) \leq_{\text{qiK}}^l F(x_0)$.*
- (b) *If $F(x_0) \leq_{\text{qiK}}^l -Tw + \Phi_C^{sv}(x, w)$, then $-Tw + \Phi_C^{sv}(x, w) \sim^l F(x_0)$.*

Proof. (a) We start by noticing that $F(x_0) = \Phi_C^{sv}(x_0, 0) = -T0 + \Phi_C^{sv}(x_0, 0)$. Hence, we have from the hypothesis that

$$-T0 + \Phi_C^{sv}(x_0, 0) \leq_{\text{qiK}}^l -Tw + \Phi_C^{sv}(x, w),$$

which is equivalent to

$$-(0, T)(x_0, 0) + \Phi_C^{sv}(x_0, 0) \leq_{\text{qiK}}^l -(0, T)(x, w) + \Phi_C^{sv}(x, w).$$

In order to apply Theorem 2.3 (a), for the function Φ_C^{sv} and the linear continuous operator $(0, T) \in \mathcal{L}(X, Y) \times \mathcal{L}(W, Y)$, we still need to verify (2.10). However, we know that, in this case, (2.10) is equivalent to $(0, T) \in \partial_{\text{qiK}} \Phi_C^{sv}(x, w)$, which is true, due to the fact that $(T, x, w) \in \mathcal{A}_{CD_{\text{qi}}^{sv}}$. Thus, applying Theorem 2.3 (a), we get

$$-(0, T)(x, w) + \Phi_C^{sv}(x, w) \leq_{\text{qiK}}^l -(0, T)(x_0, 0) + \Phi_C^{sv}(x_0, 0),$$

which means that

$$-Tw + \Phi_C^{sv}(x, w) \leq_{\text{qiK}}^l F(x_0).$$

(b) It can be proved in a similar manner, but using this time Theorem 2.3 (b) instead of Theorem 2.3 (a). ■

The forthcoming result contains some **optimality conditions** for the primal-dual pair $(P_{\text{qi}}^{sv}, D_{\text{qi}}^{sv})$ of set-valued optimization problems.

Theorem 3.2 *Let the hypotheses (3.1) be satisfied, let $\bar{x} \in \mathcal{A}_{CP_{\text{qi}}^{sv}}$, and let $(\tilde{T}, \tilde{x}, \tilde{w}) \in \mathcal{A}_{CD_{\text{qi}}^{sv}}$ be such that*

$$(3.4) \quad F(\bar{x}) \leq_{\text{qiK}}^l -\tilde{T}\tilde{w} + \Phi_C^{sv}(\tilde{x}, \tilde{w}).$$

Then the following statements are true:

- (a) $\bar{x}$ is a qi-efficient solution to (CP_{qi}^{sv}) .
- (b) $\tilde{T}$ is a qi-efficient solution to (CD_{qi}^{sv}) .

Proof. (a) Let $x \in \mathcal{A}_{CD_{\text{qi}}^{sv}}$ be such that $F(x) \leq_{\text{qiK}}^l F(\bar{x})$. Taking into account that the relation $\leq_{\text{qiK}}^l$ is transitive, we get $F(x) \leq_{\text{qiK}}^l -\tilde{T}\tilde{w} + \Phi_C^{sv}(\tilde{x}, \tilde{w})$. From Theorem 3.1 (a) it follows that $-\tilde{T}\tilde{w} + \Phi_C^{sv}(\tilde{x}, \tilde{w}) \leq_{\text{qiK}}^l F(x)$. Using again the transitivity of the relation $\leq_{\text{qiK}}^l$ and (3.4) we get that $F(\bar{x}) \leq_{\text{qiK}}^l F(x)$, which means that $\bar{x}$ is a qi-efficient solution to (CP_{qi}^{sv}) .

(b) Let us consider $(T, x, w) \in \mathcal{A}_{D_{\text{qi}}^{sv}}$ such that

$$-\tilde{T}\tilde{w} + \Phi_C^{sv}(\tilde{x}, \tilde{w}) \leq_{\text{qiK}}^l -Tw + \Phi_C^{sv}(x, w).$$

Using the transitivity of the relation $\leq_{\text{qiK}}^l$ and (3.4), we get that $F(\bar{x}) \leq_{\text{qiK}}^l -Tw + \Phi_C^{sv}(x, w)$, which, from Theorem 3.1 (a) implies $-Tw + \Phi_C^{sv}(x, w) \leq_{\text{qiK}}^l F(\bar{x})$. Applying again (3.4), we reach the conclusion

$$-Tw + \Phi_C^{sv}(x, w) \leq_{\text{qiK}}^l -\tilde{T}\tilde{w} + \Phi_C^{sv}(\tilde{x}, \tilde{w}),$$

which means that $(\tilde{T}, \tilde{x}, \tilde{w}) \in 1 - \text{Max}_{\text{qi}}\{-Tw + \Phi_C^{sv}(x, w) : (T, x, w) \in \mathcal{A}_{CD_{\text{qi}}^{sv}}\}$. Thus the conditions (3.3) holds. Moreover, from the hypotheses we know that (3.2) is also satisfied, implying that $\tilde{T}$ is a qi-efficient solution to the dual problem (CD_{qi}^{sv}) . ■

Theorem 3.3 *Let the hypotheses (3.1) be satisfied, and let $\bar{x} \in \mathcal{A}_{CP_{\text{qi}}^{sv}}$. If there exists an operator $\bar{T} \in \mathcal{L}(W, Y)$ such that $(\bar{T}, \bar{x}, 0) \in \mathcal{A}_{CD_{\text{qi}}^{sv}}$, then $\bar{T}$ is a qi-efficient solution to (CD_{qi}^{sv}) .*

Proof. Let us start by noticing that, by assuming that $(\bar{T}, \bar{x}, 0) \in \mathcal{A}_{CD_{\text{qi}}^{sv}}$, the condition (3.2) in Definition 3.1 is satisfied. Let now $(T, x, w) \in \mathcal{A}_{CD_{\text{qi}}^{sv}}$ be such that

$$(3.5) \quad -\bar{T}0 + \Phi_C^{sv}(\bar{x}, 0) \leq_{\text{qiK}}^l -Tw + \Phi_C^{sv}(x, w).$$

As Φ_C^{sv} is a perturbation function associated with F we have, $\Phi_C^{sv}(\bar{x}, 0) = F(\bar{x})$. The operator T is linear, therefore $T0 = 0$. Hence (3.5) turns into

$$(3.6) \quad F(\bar{x}) \leq_{\text{qiK}}^l -Tw + \Phi_C^{sv}(x, w),$$

which implies, by Theorem 3.1(a) that

$$-Tw + \Phi_C^{sv}(x, w) \leq_{\text{qiK}}^l F(\bar{x}) = -\bar{T}0 + \Phi_C^{sv}(\bar{x}, 0).$$

Therefore it holds $-\bar{T}0 + \Phi_C^{sv}(\bar{x}, 0) \in 1 - \text{Max}_{\text{qi}}\{-Tw + \Phi_C^{sv}(x, w) : (T, x, w) \in \mathcal{A}_{CD_{\text{qi}}^{sv}}\}$, i.e. the condition (3.3) is satisfied. Thus $\bar{T}$ is a qi-efficient solution to the dual problem (CD_{rmqi}^{sv}) . ■

Remark 3.1 *Let the hypotheses of Theorem 3.3 be satisfied. Then each operator $\tilde{T} \in \mathcal{L}(W, Y)$ for which there exists an $(\tilde{x}, \tilde{w}) \in \text{dom}\Phi_C^{sv}$ such that the following conditions are valid:*

$$(\tilde{T}, \tilde{x}, \tilde{w}) \in \mathcal{A}_{LCD_{\text{qi}}^{sv}} \text{ and } F(\bar{x}) \leq_{\text{qiK}}^l -\tilde{T}\tilde{w} + \Phi_C^{sv}(\tilde{x}, \tilde{w}),$$

is a qi-efficient solution to (CD_{qi}^{sv}) . The proof follows easily by applying Theorem 3.2 ■

4 An Application to a Set-Valued Optimization Problem in $\ell^2(\mathbb{R})$

In this section we present an example of a set-valued optimization problem for which we can apply the optimality conditions of Theorem 3.3. Consider the constrained set-valued optimization problem:

$$(P_{\ell^2(\mathbb{R})}^{sv}) \quad 1 - \text{Min}_{\text{qi}} \quad \left\{ \|\mu\|_{\ell^2(\mathbb{R})} + \|\eta\|_{\ell^2(\mathbb{R})} \right\} .,$$

$$\mu, \eta \in \ell_+^2(\mathbb{R})$$

$$G(\mu, \eta) \cap -\ell_+^2(\mathbb{R}) \neq \emptyset$$

where the objective function is $F : \ell^2(\mathbb{R}) \times \ell^2(\mathbb{R}) \rightarrow \mathcal{P}(\mathbb{R})$ is defined by

$$F(\mu) := \begin{cases} \left\{ \|\mu\|_{\ell^2(\mathbb{R})} + \|\eta\|_{\ell^2(\mathbb{R})} \right\} & \text{if } \mu, \eta \in \ell_+^2(\mathbb{R}) \\ \emptyset & \text{otherwise} \end{cases} .$$

The function containing the constraints is $G : \ell^2(\mathbb{R}) \times \ell^2(\mathbb{R}) \rightarrow \mathcal{P}(\ell^2(\mathbb{R}))$, defined by

$$G(\mu, \eta) := \begin{cases} \{-\mu - \eta\} & \text{if } \mu, \eta \in \ell_+^2(\mathbb{R}) \\ \emptyset & \text{otherwise} \end{cases}.$$

We notice that $(\text{dom}F) \cap (\text{dom}G) = \ell_+^2(\mathbb{R}) \times \ell_+^2(\mathbb{R})$ and $\text{qi}K = \text{qi}\mathbb{R}_+$. As $\mathbb{R}$ is a finite dimensional space, we have $\text{qi}\mathbb{R}_+ = \text{int}\mathbb{R}_+ = (0, +\infty)$.

Let us recall some important properties concerning the set

$$\ell^2(\mathbb{R}) := \{\mu : \mathbb{R} \rightarrow \mathbb{R} : \sum_{x \in \mathbb{R}} |\mu(x)|^2 < +\infty\}.$$

The function $\|\cdot\|_{\ell^2(\mathbb{R})} : \ell^2(\mathbb{R}) \rightarrow \mathbb{R}$ defined by

$$\|\mu\|_{\ell^2(\mathbb{R})} := \left(\sum_{x \in \mathbb{R}} |\mu(x)|^2 \right)^{\frac{1}{2}} = \left(\sup_{F \in \mathcal{P}_0(\mathbb{R}), F \text{ finite}} \sum_{x \in F} |\mu(x)|^2 \right)^{\frac{1}{2}}$$

for all $\mu \in \ell^2(\mathbb{R})$ is a norm on $\ell^2(\mathbb{R})$, and the vector space $\ell^2(\mathbb{R})$, equipped with this norm is a Banach space. The dual space $(\ell^2(\mathbb{R}))^*$ is identified with $\ell^2(\mathbb{R})$. Moreover, the set

$$\ell_+^2(\mathbb{R}) := \{\mu \in \ell^2(\mathbb{R}) : \mu(x) \geq 0 \text{ for all } x \in \mathbb{R}\}$$

is a pointed convex cone. Furthermore, it holds

$$(4.1) \quad \ell_+^2(\mathbb{R}) - \ell_+^2(\mathbb{R}) = \ell^2(\mathbb{R}).$$

We associate with $(P_{\ell^2(\mathbb{R})}^{sv})$ a Lagrange-type set-valued dual problem, with the help of the perturbation function $\Phi_{\ell^2(\mathbb{R})} : \ell^2(\mathbb{R}) \times \ell^2(\mathbb{R}) \times \ell^2(\mathbb{R}) \rightarrow \mathbb{R}$ defined by

$$\begin{aligned} \Phi_{\ell^2(\mathbb{R})}(\mu, \eta, \zeta) &:= \{F(\mu, \eta) : \mu, \eta \in \ell^2(\mathbb{R}), G(\mu, \eta) - \zeta \in -\ell_+^2(\mathbb{R})\} \\ &= \{\|\mu\|_{\ell^2(\mathbb{R})} + \|\eta\|_{\ell^2(\mathbb{R})} : \mu, \eta \in \ell_+^2(\mathbb{R}), \zeta \in -\mu - \eta + \ell_+^2(\mathbb{R})\}, \end{aligned}$$

for all $(\mu, \eta, \zeta) \in \ell^2(\mathbb{R}) \times \ell^2(\mathbb{R}) \times \ell^2(\mathbb{R})$.

The qi-conjugate set-valued function associated with $\Phi_{\ell^2(\mathbb{R})}$ is

$$(\Phi_{\ell^2(\mathbb{R})})_{\text{qi}\mathbb{R}_+}^* : \mathcal{L}(\ell^2(\mathbb{R}), \mathbb{R}) \times \mathcal{L}(\ell^2(\mathbb{R}), \mathbb{R}) \times \mathcal{L}(\ell^2(\mathbb{R}), \mathbb{R}) \rightarrow \mathcal{P}(\mathcal{P}(\mathbb{R})),$$

defined by

$$(\Phi_{\ell^2(\mathbb{R})})_{\text{qi}\mathbb{R}_+}^*(0, T) := \text{u} - \text{Max}_{\text{qi}} \left\{ \begin{array}{l} T\zeta - \Phi(\mu, \eta, \zeta) : \\ (\mu, \eta, \zeta) \in \ell^2(\mathbb{R}) \times \ell^2(\mathbb{R}) \times \ell^2(\mathbb{R}) \end{array} \right\}$$

for all $T \in \mathcal{L}(\ell^2(\mathbb{R}), \mathbb{R})$.

A Lagrange-type set-valued dual problem associated with $(P_{\ell^2(\mathbb{R})}^{sv})$ is

$$(LD_{\ell^2(\mathbb{R})}^{sv}) \quad 1 - \text{Max}_{\text{qi}} \left\{ T \in \mathcal{L}(\ell^2(\mathbb{R}), \mathbb{R}) \left[-(\Phi_{\ell^2(\mathbb{R})})_{\text{qi}\mathbb{R}_+}^*(0, T) \right] \right\}.$$

In the following we use the notation

$$\mathcal{A}_{LD_{\ell^2(\mathbb{R})}^{sv}} := \left\{ (T, \mu, \eta, \zeta) : \begin{array}{l} T \in \mathcal{L}(\ell^2(\mathbb{R}), \mathbb{R}), \mu, \eta, \zeta \in \ell^2(\mathbb{R}), \\ T\zeta - \Phi(\mu, \eta, \zeta) \in -(\Phi_{\ell^2(\mathbb{R})})_{\text{qi}\mathbb{R}_+}^*(0, T) \end{array} \right\}.$$

Let us further notice that

$$\mathcal{A}_{LD_{\ell^2(\mathbb{R})}^{sv}} = \left\{ (T, \mu, \eta, \zeta) : \begin{array}{l} T \in \mathcal{L}(\ell^2(\mathbb{R}), \mathbb{R}), \zeta \in -\mu - \eta + \ell_+^2(\mathbb{R}), \\ \mu, \eta \in \ell_+^2(\mathbb{R}), (0, T) \in \partial_{\text{qi}\mathbb{R}_+}(\Phi_{\ell^2(\mathbb{R})})(\mu, \zeta) \end{array} \right\}.$$

An operator $\tilde{T} \in \mathcal{L}(\ell^2(\mathbb{R}), \mathbb{R})$ is a **qi-efficient solution** to $(LD_{\ell^2(\mathbb{R})}^{sv})$ if there exists an $(\tilde{\mu}, \tilde{\eta}, \tilde{\zeta}) \in \text{dom} \Phi_{\ell^2(\mathbb{R})}$ such that

$$(\tilde{T}, \tilde{\mu}, \tilde{\eta}, \tilde{\zeta}) \in \mathcal{A}_{LD_{\ell^2(\mathbb{R})}^{sv}}$$

and

$$-\tilde{T}\tilde{\zeta} + \|\tilde{\mu}\|_{\ell^2(\mathbb{R})} + \|\tilde{\eta}\|_{\ell^2(\mathbb{R})} \in 1 - \text{Max}_{\text{qi}} \left\{ \begin{array}{l} -T\zeta + \|\mu\|_{\ell^2(\mathbb{R})} + \|\eta\|_{\ell^2(\mathbb{R})} : \\ (T, \mu, \eta, \zeta) \in \mathcal{A}_{LD_{\ell^2(\mathbb{R})}^{sv}} \end{array} \right\}.$$

We notice that, $(\bar{\mu}, \bar{\eta}) = (0, 0)$ is a qi-efficient solution to $(P_{\ell^2(\mathbb{R})}^{sv})$. Moreover, (F, G) is a $\mathbb{R}_+ \times \ell_+^2(\mathbb{R})$ -convex function. Next, from (4.1) it

follows that

$$(4.2) \quad 0 \in \text{qi} \left[-\ell_+^2(\mathbb{R}) + \ell_+^2(\mathbb{R}) \right] = \text{qi} \ell^2(\mathbb{R}) = \ell^2(\mathbb{R}).$$

We prove that

$$(4.3) \quad (0, 0) \notin \text{qri} \left[(F, G) (\ell^2(\mathbb{R}) \times \ell^2(\mathbb{R})) + \mathbb{R}_+ \times \ell_+^2(\mathbb{R}) \right].$$

By contradiction we prove that (4.3) is satisfied. Assume that

$$(0, 0) \in \text{qri} \left[(F, G) (\ell^2(\mathbb{R}) \times \ell^2(\mathbb{R})) + \mathbb{R}_+ \times \ell_+^2(\mathbb{R}) \right]$$

and consider an arbitrary $(r, \mu^*) \in N_{(F, G) (\ell^2(\mathbb{R}) \times \ell^2(\mathbb{R})) + \mathbb{R}_+ \times \ell_+^2(\mathbb{R})} \{(0, 0)\}$. Then for all $\mu, \eta \in \ell_+^2(\mathbb{R}) \times \ell_+^2(\mathbb{R})$, all $\lambda \geq 0$, and all $\chi \in \ell_+^2(\mathbb{R})$ it holds

$$(4.4) \quad r(\|\mu\|_{\ell^2(\mathbb{R})} + \|\eta\|_{\ell^2(\mathbb{R})} + \lambda) + \langle \mu^*, -\mu - \eta + \chi \rangle_{\ell^2(\mathbb{R})} \leq 0.$$

We notice that

$$(-1, 0) \in N_{(F, G) (\ell^2(\mathbb{R}) \times \ell^2(\mathbb{R})) + \mathbb{R}_+ \times \ell_+^2(\mathbb{R})} \{(0, 0)\}.$$

By applying Theorem 2.1 we get

$$(1, 0) \in N_{(F, G) (\ell^2(\mathbb{R}) \times \ell^2(\mathbb{R})) + \mathbb{R}_+ \times \ell_+^2(\mathbb{R})} \{(0, 0)\}.$$

This means that (4.4) turns into

$$\|\mu\|_{\ell^2(\mathbb{R})} + \|\eta\|_{\ell^2(\mathbb{R})} + \lambda \leq 0 \text{ for all } \mu, \eta \in \ell_+^2(\mathbb{R}) \text{ and all } \lambda \geq 0,$$

which is obviously a contradiction. Hence (4.3) is satisfied. From (4.3) and the fact that the set $(F, G) (\ell^2(\mathbb{R}) \times \ell^2(\mathbb{R})) + \mathbb{R}_+ \times \ell_+^2(\mathbb{R})$ is $\mathbb{R}_+ \times \ell_+^2(\mathbb{R})$ -convex we may apply a separation theorem involving the quasi-relative interior (see Theorem 2.3 in R. I. BOT, E. R. CSETNEK, G. WANKA [3]). Thus, there exist

$$(4.5) \quad (y^*, \zeta^*) \in \mathbb{R} \times \ell^2(\mathbb{R}) \setminus \{(0, 0)\}$$

such that

$$(4.6) \quad 0 \leq y^* \cdot y + \langle \zeta^*, \zeta \rangle$$

for all $(y, \zeta) \in (F, G)(\ell^2(\mathbb{R}) \times \ell^2(\mathbb{R})) + \mathbb{R}_+ \times \ell_+^2(\mathbb{R})$.

We proceed by proving that

$$(4.7) \quad y^* \geq 0.$$

It is clear that $0 \in G(0, 0) + \ell_+^2(\mathbb{R})$. Putting $\mu = \eta = 0$ and $\zeta = 0$ in (4.6), we obtain

$$0 \leq y^* \cdot y \text{ for all } y \in F(0, 0) + \mathbb{R}_+ = \mathbb{R}_+.$$

Therefore (4.7) holds.

Next we prove that

$$(4.8) \quad \zeta^* \in \ell_+^2(\mathbb{R}).$$

Put $(\mu, \eta) := (0, 0)$ and $y = 0$ in (4.6). Then we get

$$(4.9) \quad 0 \leq \langle \zeta^*, \zeta \rangle \text{ for all } \zeta \in G(0, 0) + \ell_+^2(\mathbb{R}) = \ell_+^2(\mathbb{R}).$$

Hence (4.8) holds.

We now prove that $y^* > 0$. Assume that $y^* = 0$. Then (4.6) turns into

$$0 \leq \langle \zeta^*, \zeta \rangle \text{ for all } \zeta \in G(\ell^2(\mathbb{R}) \times \ell^2(\mathbb{R})) + \ell_+^2(\mathbb{R}).$$

Hence, $-\zeta^* \in N_{G(\ell^2(\mathbb{R}) \times \ell^2(\mathbb{R})) + \ell_+^2(\mathbb{R})}(0)$. According to Theorem 2.2, (4.2) implies

$$N_{G(\ell^2(\mathbb{R}) \times \ell^2(\mathbb{R})) + \ell_+^2(\mathbb{R})} = \{0\},$$

therefore $-\zeta^* = 0$. But in this case we have $(y^*, \zeta^*) = (0, 0)$, which contradicts (4.5). Thus $y^* \neq 0$, and from (4.7) we obtain

$$(4.10) \quad y^* > 0.$$

We define now the operator $\bar{T} : \ell^2(\mathbb{R}) \rightarrow \mathbb{R}$ by

$$\bar{T}(\zeta) := -\frac{\langle \zeta^*, \zeta \rangle}{y^*} \text{ for all } \zeta \in \ell^2(\mathbb{R})$$

and we prove that

$$(4.11) \quad (\bar{T}, 0, 0, 0) \in \mathcal{A}_{LD_{\ell^2(\mathbb{R})}^{sv}}.$$

Let us proceed by contradiction. Since all the other conditions are satisfied, we only contradict $(0, \bar{T}) \in \partial_{\text{qi}\mathbb{R}_+}(\Phi_{\ell^2(\mathbb{R})})(0, 0)$. Assume that there exist

$$(\mu, \eta) \in \ell_+^2(\mathbb{R}) \times \ell_+^2(\mathbb{R}) \text{ and } \zeta \in G(\mu, \eta) + \ell_+^2(\mathbb{R})$$

such that

$$-\bar{T}(\zeta) + F(\mu, \eta) \preceq_{\text{qi}\mathbb{R}_+} -\bar{T}(0) + F(0, 0).$$

This means that $F(0, 0) \subseteq F(\mu, \eta) - \bar{T}(\zeta) + \text{qi}\mathbb{R}_+$. Thus there exist $k \in \text{qi}\mathbb{R}_+ = (0, \infty)$ such that

$$0 = \|\mu\|_{\ell^2(\mathbb{R})} + \|\eta\|_{\ell^2(\mathbb{R})} - \bar{T}(\zeta) + k.$$

Since $y^* > 0$, by applying the definition of $\bar{T}$ we get

$$\begin{aligned} 0 &= y^* \cdot (\|\mu\|_{\ell^2(\mathbb{R})} + \|\eta\|_{\ell^2(\mathbb{R})}) + y^* \cdot \frac{\langle \zeta^*, \zeta \rangle}{y^*} + y^* \cdot k \\ &= y^* \cdot (\|\mu\|_{\ell^2(\mathbb{R})} + \|\eta\|_{\ell^2(\mathbb{R})}) + \langle \zeta^*, \zeta \rangle + y^* \cdot k. \end{aligned}$$

Using now the facts that $y^* \in \mathbb{R}^+ \setminus \{0\}$, $k \in \text{qi}\mathbb{R}_+$, and applying Proposition 2.1, it follows that $y^* \cdot k > 0$. Therefore we come to the conclusion that

$$(4.12) \quad 0 > y^* \cdot (\|\mu\|_{\ell^2(\mathbb{R})} + \|\eta\|_{\ell^2(\mathbb{R})}) + \langle \zeta^*, \zeta \rangle.$$

Considering the facts that $(\mu, \eta) \in (\text{dom}F) \cap (\text{dom}G)$, $\|\mu\|_{\ell^2(\mathbb{R})} + \|\eta\|_{\ell^2(\mathbb{R})} \in F(\mu, \eta) \subseteq F(\mu, \eta) + \mathbb{R}_+$ and $\zeta \in G(\mu, \eta) + \ell_+^2(\mathbb{R})$, it follows that (4.12) is in contradiction to (4.6). This leads to $(\bar{T}, 0, 0, 0) \in \mathcal{A}_{LD_{\ell^2(\mathbb{R})}^{sv}}$.

Thus, this is a situation in which Theorem 3.3 can be applied. Therefore, we get from (4.11) that $\bar{T}$ is a qi-efficient solution to $(LD_{\ell^2(\mathbb{R})}^{sv})$.

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