

On (h, k) -trichotomy for evolution operators in Banach spaces

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ABSTRACT. The paper considers three general trichotomy concepts, which are generalizations of the classical exponential and polynomial trichotomies. Characterizations and connections between these concepts are given.

KEY WORDS: Evolution operators, (h, k) -trichotomy, strong (h, k) -trichotomy, weak (h, k) -trichotomy

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1 Introduction

As natural generalizations of the dichotomy properties, the concepts of trichotomy introduced by R. J. Sacker and G. R. Sell in [15] play a crucial role in the qualitative theory of dynamical systems. The first studies devoted to the trichotomic behaviors for differential equations were initiated by S.

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Elaydi and O. Hayek in [4] and [5].

In the last decades, a substantial part of the asymptotic theory of differential equations was devoted to the extension of the methods used in dichotomy theory to the trichotomy case (see [7], [8], [9], [12], [13], [14]).

Characterizations for the uniform exponential trichotomy of evolution operators in Banach spaces was obtained by M. Megan and C. Stoica in [10], [11] and M. I. Kovács in [6].

In the nonautonomous setting the concept of uniform exponential (or polynomial) trichotomy is too restrictive and it is important to look for more general behaviors, for example the nonuniform case, where a consistent contribution is due to L. Barreira and C. Valls (see [1], [2], [3]).

A new perspective on the discrete input-output techniques in the detection of the trichotomic behavior of nonautonomous dynamical systems is presented by B. Sasu and A.L. Sasu in [16].

This paper considers three general trichotomy concepts with growth rates given by increasing functions. These new trichotomies include the traditional (uniform or nonuniform) exponential or polynomial trichotomy. Thus we obtain a systematic classification of trichotomy concepts with the connections between them.

2 Definitions, notations and preliminary results

Let X be a real or complex Banach space and let $\mathcal{B}(X)$ be the algebra Banach of all bounded linear operators on X . The norm on X and on $\mathcal{B}(X)$ will be denoted by $\|\cdot\|$. Denote by I the identity operator on X .

Let Δ be the set of all pairs (t, s) of real numbers with $t \geq s \geq 0$. We denote by $T = \Delta \times X$.

In this section we give some preliminary definitions.

Definition 2.1 *An operator valued function $U : \Delta \rightarrow \mathcal{B}(X)$ is said to be an evolution operator on X if*

$$(e_1) \quad U(t, t) = I \text{ for every } t \geq 0;$$

$$(e_2) \quad U(t, s)U(s, t_0) = U(t, t_0) \text{ for all } (t, s) \in \Delta \text{ and } (s, t_0) \in \Delta.$$

If the condition (e_2) holds for all $t, s, t_0 \geq 0$ then we say that U is a reversible evolution operator .

Definition 2.2 An operator valued function $P : \mathbb{R}_+ \rightarrow \mathcal{B}(X)$ is called a family of projections on X if

$$P(t)^2 = P(t), \quad \text{for every } t \geq 0.$$

Definition 2.3 Given an evolution operator $U : \Delta \rightarrow \mathcal{B}(X)$, we say that a family of projections $P : \mathbb{R}_+ \rightarrow \mathcal{B}(X)$ is invariant for U if

$$P(t)U(t, s) = U(t, s)P(s)$$

for all $(t, s) \in \Delta$.

Definition 2.4 Three families of projections $P_1, P_2, P_3 : \mathbb{R}_+ \rightarrow \mathcal{B}(X)$ are called supplementary if for every $t \in \mathbb{R}_+$ we have

$$(s_1) \quad P_1(t) + P_2(t) + P_3(t) = I;$$

$$(s_2) \quad P_i(t)P_j(t) = 0 \text{ for all } i, j \in \{1, 2, 3\} \text{ with } i \neq j.$$

If $\mathcal{P} = \{P_1, P_2, P_3\}$ is a family of three supplementary projections which are invariant for U , then we say that the pair $(U, \mathcal{P})$ is a trichotomic pair.

Definition 2.5 Let $P : \mathbb{R}_+ \rightarrow \mathcal{B}(X)$ be a family of projections on X which is invariant for the evolution operator $U : \Delta \rightarrow \mathcal{B}(X)$. We say that P is strongly invariant for U if for all $(t, s) \in \Delta$ the restriction of $U(t, s)$ on $\text{Range}P(s)$ is an isomorphism from $\text{Range}P(s)$ to $\text{Range}P(t)$.

Remark 2.1 If a family of projections $P : \mathbb{R}_+ \rightarrow \mathcal{B}(X)$ is strongly invariant for the evolution operator $U : \Delta \rightarrow \mathcal{B}(X)$ then there exists $V : \Delta \rightarrow \mathcal{B}(X)$ such that $V(t, s)$ is an isomorphism from $\text{Range}P(t)$ to $\text{Range}P(s)$ and

$$(v_1) \quad U(t, s)V(t, s)P(t) = P(t);$$

$$(v_2) \quad V(t, s)U(t, s)P(s) = P(s),$$

for all $(t, s) \in \Delta$.

The map V is called the skew-evolution operator associated to the pair (U, P) .

Definition 2.6 A family $\mathcal{P} = \{P_1, P_2, P_3\}$ of three supplementary projections is called compatible with the evolution operator $U : \Delta \rightarrow \mathcal{B}(X)$ if

- (c₁) P_1 is invariant for U ;
- (c₂) P_2 and P_3 are strongly invariant for U .

For an evolution operator $U : \Delta \rightarrow \mathcal{B}(X)$ and $\mathcal{P}$ compatible with U , we will denote by $V_2(t, s)$ and $V_3(t, s)$ the skew-evolution operators associated to the pairs (U, P_2) and (U, P_3) .

3 (h, k) -trichotomy

Let $h, k : \mathbb{R}_+ \rightarrow [1, \infty)$ be two increasing functions and $(U, \mathcal{P})$ a trichotomic pair.

Definition 3.1 We say that the pair $(U, \mathcal{P})$ is (h, k) -trichotomic and denote $(h, k) - t$, if there are the constants $N \geq 1, a > 0, b \geq 0, c > 0$ such that:

- (t₁) $h(t)^a \|U(t, s)P_1(s)x\| \leq Nh(s)^a k(s)^b \|P_1(s)x\|;$
- (t₂) $h(t)^a \|P_2(s)x\| \leq Nh(s)^a k(t)^b \|U(t, s)P_2(s)x\|;$
- (t₃) $h(s)^c \|U(t, s)P_3(s)x\| \leq Nh(t)^c k(s)^b \|P_3(s)x\|;$
- (t₄) $h(s)^c \|P_3(s)x\| \leq Nh(t)^c k(t)^b \|U(t, s)P_3(s)x\|,$

for all $(t, s, x) \in \Delta \times X$.

Remark 3.1 As particular cases of (h, k) -trichotomy we remark that

- (i) if $h(t) = k(t) = e^t$, then we recover the notion of nonuniform exponential trichotomy and in particular when the function k is constant (or $b = 0$) we obtain the classical notion of uniform exponential trichotomy.
- (ii) if $h(t) = k(t) = t + 1$, then we recover the notion of nonuniform polynomial trichotomy and in particular when the function k is constant (or $b = 0$) we obtain the classical notion of uniform polynomial trichotomy.

- (iii) if $P_3 = 0$ in Definition 3.1, then we recover the notion of (h, k) -dichotomy.
- (iv) if $h(t) = k(t) = e^t$ and $P_3 = 0$ then we recover the notion of nonuniform exponential dichotomy and in particular when the function k is constant (or $b = 0$) we obtain the classical notion of uniform exponential dichotomy.
- (v) if $h(t) = k(t) = t + 1$ and $P_3 = 0$, then we recover the notion of nonuniform polynomial dichotomy and in particular when the function k is constant (or $b = 0$) we obtain the classical notion of uniform polynomial dichotomy.

An example of a trichotomic pair $(U, \mathcal{P})$ which is (h, k) -trichotomic is given below.

Example 3.1 On $X = \mathbb{R}^3$ endowed with the norm

$$\|(x_1, x_2, x_3)\| = \max\{|x_1|, |x_2|, |x_3|\}.$$

We consider the families of projection $P_1, P_2, P_3 : \mathbb{R}_+ \rightarrow \mathcal{B}(X)$ defined by

$$P_1(t)(x_1, x_2, x_3) = (x_1, 0, 0)$$

$$P_2(t)(x_1, x_2, x_3) = (0, x_2, 0)$$

$$P_3(t)(x_1, x_2, x_3) = (0, 0, x_3)$$

for every $t \geq 0$ and $x = (x_1, x_2, x_3) \in \mathbb{R}^3$.

Given the increasing functions $h, k : \mathbb{R}_+ \rightarrow [1, \infty)$ we consider the evolution operator $U : \Delta \rightarrow \mathcal{B}(X)$ defined by

$$U(t, s)(x_1, x_2, x_3) = \left(\frac{k(s)}{k(t)} \cdot \left(\frac{h(s)}{h(t)} \right)^2 x_1, \frac{k(t)}{k(s)} \cdot \left(\frac{h(t)}{h(s)} \right)^2 x_2, \frac{h(s)}{h(t)} x_3 \right)$$

So

$$U(t, s) = \frac{k(s)}{k(t)} \cdot \left(\frac{h(s)}{h(t)} \right)^2 P_1(s) + \frac{k(t)}{k(s)} \cdot \left(\frac{h(t)}{h(s)} \right)^2 P_2(s) + \frac{h(s)}{h(t)} P_3(s)$$

and hence

$$h(t)^2 \|U(t, s)P_1(s)x\| = \frac{k(s)}{k(t)} \cdot h(s)^2 \|P_1(s)x\| \leq k(s) \cdot h(s)^2 \|P_1(s)x\|,$$

$$h(t)^2 \|P_2(s)x\| = \frac{k(s)}{k(t)} \cdot h(s)^2 \|U(t, s)P_2(s)x\| \leq k(t)h(s)^2 \|U(t, s)P_2(s)x\|,$$

$$h(s) \|U(t, s)P_3(s)x\| = \frac{h(s)^2}{h(t)} \|P_3(s)x\| \leq h(s) \|P_3(s)x\| \leq h(t)k(s) \|P_3(s)x\|,$$

$$h(s) \|P_3(s)x\| = h(t) \|U(t, s)P_3(s)x\| \leq h(t)k(t) \|U(t, s)P_3(s)x\|,$$

for all $(t, s, x) \in \Delta \times X$.

So for all (h, k) there exists an evolution operator U such that U is (h, k) -trichotomic.

A first characterization of (h, k) -trichotomy is given by

Theorem 3.1 *The trichotomic pair $(U, \mathcal{P})$ is (h, k) -trichotomic if and only if there are $N \geq 1, a > 0, b \geq 0, c > 0$*

$$(t'_1) \quad h(t)^a \|U(t, t_0)P_1(t_0)x_0\| \leq Nh(s)^a k(s)^b \|U(s, t_0)P_1(t_0)x_0\|;$$

$$(t'_2) \quad h(t)^a \|U(s, t_0)P_2(t_0)x_0\| \leq Nh(s)^a k(t)^b \|U(t, t_0)P_2(t_0)x_0\|;$$

$$(t'_3) \quad h(s)^c \|U(t, t_0)P_3(t_0)x_0\| \leq Nh(t)^c k(s)^b \|U(s, t_0)P_3(t_0)x_0\|;$$

$$(t'_4) \quad h(s)^c \|U(s, t_0)P_3(t_0)x_0\| \leq Nh(t)^c k(t)^b \|U(t, t_0)P_3(t_0)x_0\|,$$

for all $t \geq s \geq t_0 \geq 0$ and $x_0 \in X$.

Proof. Necessity.

$(t_1) \Rightarrow (t'_1)$ If we suppose that (t_1) holds then

$$\begin{aligned} h(t)^a \|U(t, t_0)P_1(t_0)x_0\| &= h(t)^a \|U(t, s)P_1(s)U(s, t_0)P_1(t_0)x_0\| \leq \\ &\leq Nh(s)^a k(s)^b \|P_1(s)U(s, t_0)P_1(t_0)x_0\| = \\ &= Nh(s)^a k(s)^b \|U(s, t_0)P_1(t_0)x_0\|, \end{aligned}$$

for all $(t, s, t_0, x_0) \in \mathbb{R}_+^3 \times X$ with $t \geq s \geq t_0$.

$(t_2) \Rightarrow (t'_2)$ From (t_2) it follows

$$\begin{aligned} h(t)^a \|U(s, t_0)P_2(t_0)x_0\| &= h(t)^a \|P_2(s)U(s, t_0)P_2(t_0)x_0\| \leq \\ &\leq Nh(s)^a k(t)^b \|U(t, s)P_2(s)U(s, t_0)P_2(t_0)x_0\| = \\ &= Nh(s)^a k(t)^b \|U(t, t_0)P_2(t_0)x_0\|, \end{aligned}$$

and hence (t'_2) is verified.

$(t_3) \Rightarrow (t'_3)$ By (t_3) we have

$$\begin{aligned} h(s)^c \|U(t, t_0)P_3(t_0)x_0\| &= h(s)^c \|U(t, s)P_3(s)U(s, t_0)P_3(t_0)x_0\| \leq \\ &\leq Nh(t)^c k(s)^b \|P_3(s)U(s, t_0)P_3(t_0)x_0\| = Nh(t)^c k(s)^b \|U(s, t_0)P_3(t_0)x_0\|, \end{aligned}$$

for all $t \geq s \geq t_0 \geq 0$ and all $x_0 \in X$.

$(t_4) \Rightarrow (t'_4)$ The inequality (t_4) implies

$$\begin{aligned} h(s)^c \|U(s, t_0)P_3(t_0)x_0\| &= h(s)^c \|P_3(s)U(s, t_0)x_0\| \leq \\ &\leq Nh(t)^c k(t)^b \|U(t, s)P_3(s)U(s, t_0)x_0\| = Nh(t)^c k(t)^b \|U(t, t_0)P_3(t_0)x_0\|, \end{aligned}$$

for all $t \geq s \geq t_0$ and $x_0 \in X$, which shows that (t'_4) holds.

Sufficiency. It is obvious, taking $t_0 = s$ in (t_i) with $i \in \{1, 2, 3, 4\}$. ■

A characterization of (h, k) -trichotomy in the particular case when the family $\mathcal{P}$ is supplementary and compatible for U is given by

Theorem 3.2 *If $\mathcal{P}$ is a supplementary family of projections which is compatible for U then the pair $(U, \mathcal{P})$ is (h, k) -trichotomic if and only if there exist $N \geq 1, a > 0, b \geq 0, c > 0$ such that:*

$$(t''_1) \quad h(t)^a \|U(t, s)P_1(s)x\| \leq Nh(s)^a k(s)^b \|P_1(s)x\|;$$

$$\begin{aligned}
(t_2'') \quad & h(t)^a \|V(t, s)P_2(t)x\| \leq Nh(s)^a k(t)^b \|P_2(t)x\|; \\
(t_3'') \quad & h(s)^c \|U(t, s)P_3(s)x\| \leq Nh(t)^c k(s)^b \|P_3(s)x\|; \\
(t_4'') \quad & h(s)^c \|V(t, s)P_3(t)x\| \leq Nh(t)^c k(t)^b \|P_3(t)x\|, \\
& \text{for all } (t, s, x) \in \Delta \times X.
\end{aligned}$$

Proof. It is sufficient to prove the equivalences $(t_2) \Leftrightarrow (t_2'')$ and $(t_4) \Leftrightarrow (t_4'')$.

Necessity.

$(t_2) \Rightarrow (t_2'')$ By (t_2) we have

$$\begin{aligned}
& h(t)^a \|V(t, s)P_2(t)x\| = h(t)^a \|P_2(s)V(t, s)P_2(t)x\| \leq \\
& \leq Nh(s)^a k(t)^b \|P_2(t)U(t, s)P_2(s)V(t, s)P_2(t)x\| = Nh(s)^a k(t)^b \|P_2(t)x\|,
\end{aligned}$$

and hence (t_2'') holds.

$(t_4) \Rightarrow (t_4'')$ From (t_4) we have

$$\begin{aligned}
& h(s)^c \|V(t, s)P_3(t)x\| = h(s)^c \|P_3(s)V(t, s)P_3(t)x\| \leq \\
& \leq Nh(t)^a k(t)^b \|U(t, s)P_3(s)V(t, s)P_3(t)x\| = \\
& = Nh(t)^a k(t)^b \|P_3(t)U(t, s)V(t, s)P_3(t)x\| = Nh(t)^c k(t)^b \|P_3(t)x\|,
\end{aligned}$$

for all $(t, s, x) \in \Delta \times X$ and hence (t_4'') is proved.

Sufficiency.

$(t_2'') \Rightarrow (t_2)$ If we suppose (t_2'') then

$$h(t)^a \|P_2(s)x\| = h(t)^a \|V(t, s)U(t, s)P_2(s)x\| \leq Nh(s)^a k(t)^b \|U(t, s)P_2(s)x\|$$

and hence (t_2) is proved.

$(t_4'') \Rightarrow (t_4)$ By (t_4'') it results

$$h(s)^c \|P_3(s)x\| = h(s)^c \|V(t, s)U(t, s)P_3(s)x\| \leq Nh(t)^c k(t)^b \|U(t, s)P_3(s)x\|,$$

for all $(t, s, x) \in \Delta \times X$ and hence (t_4) holds.

■

4 Strong (h,k)-trichotomy

Now we introduce a new concept of (h,k)-trichotomy.

Definition 4.1 Let $\mathcal{P} = \{P_1, P_2, P_3\}$ be a family of three supplementary projections which is compatible with the evolution operator $U : \Delta \rightarrow \mathcal{B}(X)$. We say that the pair $(U, \mathcal{P})$ is strongly (h,k)-trichotomic and denote $s - (h, k) - t$, if there are $N \geq 1, a > 0, b \geq 0, c > 0$ such that for all $(t, s, x) \in \Delta \times X$ the following properties hold:

$$(st_1) \quad h(t)^a \|U(t, s)P_1(s)x\| \leq Nh(s)^a k(s)^b \|x\|;$$

$$(st_2) \quad h(t)^a \|V(t, s)P_2(t)x\| \leq Nh(s)^a k(t)^b \|x\|;$$

$$(st_3) \quad h(s)^c \|U(t, s)P_3(s)x\| \leq Nh(t)^c k(s)^b \|x\|;$$

$$(st_4) \quad h(s)^c \|V(t, s)P_3(t)x\| \leq Nh(t)^c k(t)^b \|x\|.$$

Remark 4.1 If the function k is constant or $b = 0$ then we obtain the concept of strong h -trichotomy.

Remark 4.2 The pair $(U, \mathcal{P})$ is strongly (h,k)-trichotomic if and only if there are $N \geq 1, a > 0, b \geq 0, c > 0$ such that:

$$(st'_1) \quad h(t)^a \|U(t, s)P_1(s)\| \leq Nh(s)^a k(s)^b;$$

$$(st'_2) \quad h(t)^a \|V(t, s)P_2(t)\| \leq Nh(s)^a k(t)^b;$$

$$(st'_3) \quad h(s)^c \|U(t, s)P_3(s)\| \leq Nh(t)^c k(s)^b;$$

$$(st'_4) \quad h(s)^c \|V(t, s)P_3(t)\| \leq Nh(t)^c k(t)^b$$

for all $(t, s) \in \Delta$.

Definition 4.2 A family of projections $\mathcal{P} = \{P_1, P_2, P_3\}$ is called k -bounded if there exists $M \geq 1$ and $d \geq 0$ such that

$$\|P_j(t)\| \leq Mk(t)^d,$$

for all $t \geq 0$ and all $j \in \{1, 2, 3\}$.

Remark 4.3 *If the pair $(U, \mathcal{P})$ is strongly (h, k) -trichotomic then $\mathcal{P}$ is k -bounded.*

Remark 4.4 *If the pair $(U, \mathcal{P})$ is strongly (h, k) -trichotomic then it is (h, k) -trichotomic.*

Proof. *Indeed, if we substitute x with $P_1(s)x$ in (st_1) , x with $P_2(t)x$ in (st_2) , x with $P_3(s)x$ in (st_3) respectively x with $P_3(t)x$ in (st_4) then we obtain that the conditions $(t_1), (t_2), (t_3)$ and (t_4) are satisfied. ■*

Example 4.1 *On $X = \mathbb{R}^3$ endowed with the norm*

$$\|(x_1, x_2, x_3)\| = \max\{|x_1|, |x_2|, |x_3|\}.$$

Let $h, k : \mathbb{R}_+ \rightarrow [1, \infty)$ be two increasing functions .

We consider the families of projections give by $\mathcal{P} = \{P_1, P_2, P_3\}$ with $P_1, P_2, P_3 : \mathbb{R}_+ \rightarrow \mathcal{B}(X)$ defined by

$$P_1(t)(x_1, x_2, x_3) = (x_1 + e^{k^2(t)}x_2, 0, 0)$$

$$P_2(t)(x_1, x_2, x_3) = (-e^{k^2(t)}x_2, x_2, 0)$$

$$P_3(t)(x_1, x_2, x_3) = (0, 0, x_3)$$

for every $t \geq 0$ and $x = (x_1, x_2, x_3) \in \mathbb{R}^3$.

Consider the evolution operator $U : \Delta \rightarrow \mathcal{B}(X)$ defined, for all $(t, s) \in \Delta$ by

$$U(t, s) = \frac{h(s)}{h(t)}P_1(s) + \frac{h(t)}{h(s)}P_2(t) + \frac{h(s)}{h(t)}P_3(s)$$

We have that,

$$h(t)\|U(t, s)P_1(s)x\| = h(s)\|P_1(s)x\| \leq k(s)h(s)\|P_1(s)x\|,$$

$$h(t)\|P_2(s)x\| \leq h(s)\|U(t, s)P_2(s)x\| \leq k(t)h(s)\|U(t, s)P_2(s)x\|,$$

$$h(s)\|U(t, s)P_3(s)x\| \leq h(s)\|P_3(s)x\| \leq h(t)k(s)\|P_3(s)x\|,$$

$$h(s)\|P_3(s)x\| = h(t)\|U(t,s)P_3(s)x\| \leq h(t)k(t)\|U(t,s)P_3(s)x\|,$$

for all $(t, s, x) \in \Delta \times X$.

So $(U, \mathcal{P})$ is (h, k) -trichotomic.

We observe that

$$\|P_1(t)(0, 1, 0)\| = e^{k^2(t)}$$

and by on Remark 4.3 it results that $(U, \mathcal{P})$ cannot be strongly (h, k) -trichotomic.

Remark 4.5 The previous example shows that for every two increasing functions $h, k : \mathbb{R}_+ \rightarrow [1, \infty)$ there exists a family of supplementary projections $P = \{P_1, P_2, P_3\}$ and an evolution operator U such that

- (i) $\mathcal{P}$ is compatible with U ;
- (ii) $(U, \mathcal{P})$ is (h, k) – trichotomic;
- (iii) $(U, \mathcal{P})$ is not strongly (h, k) – trichotomic.

5 Weak (h, k) -trichotomy

We introduce a new concept of (h, k) -trichotomy.

Definition 5.1 Let $\mathcal{P} = \{P_1, P_2, P_3\}$ be a family of three supplementary projections which is compatible with the evolution operator $U : \Delta \rightarrow \mathcal{B}(X)$. We say that the pair $(U, \mathcal{P})$ is weakly (h, k) -trichotomic and denote $w - (h, k) - t$, if there are $N \geq 1, a > 0, b \geq 0, c > 0$ such that for all $(t, s) \in \Delta$ the following properties hold:

- (wt_1) $h(t)^a \|U(t, s)P_1(s)\| \leq Nh(s)^a k(s)^b \|P_1(s)\|;$
- (wt_2) $h(t)^a \|V(t, s)P_2(t)\| \leq Nh(s)^a k(t)^b \|P_2(t)\|;$
- (wt_3) $h(s)^c \|U(t, s)P_3(s)\| \leq Nh(t)^c k(s)^b \|P_3(s)\|;$
- (wt_4) $h(s)^c \|V(t, s)P_3(t)\| \leq Nh(t)^c k(t)^b \|P_3(t)\|.$

Remark 5.1 If the pair $(U, \mathcal{P})$ is (h, k) -trichotomic then it is weakly (h, k) -trichotomic.

Remark 5.2 *The connections between the trichotomy concepts considered in this paper are*

$$s - (h, t) - t \implies (h, k) - t \implies w - (h, k) - t$$

Remark 5.3 *If $\mathcal{P} = \{P_1, P_2, P_3\}$ is k -bounded then $w - (h, k) - t$ implies $s - (h, k) - t$.*

A particular case when the trichotomy concepts considered in this paper are equivalent is given by

Remark 5.4 *Let $\mathcal{P} = \{P_1, P_2, P_3\}$ be a family of three supplementary projections which is k -bounded and compatible with the evolution operator $U : \Delta \rightarrow \mathcal{B}(X)$. Then the following statements are equivalent:*

- (i) $(U, \mathcal{P})$ is strongly (h, k) -trichotomic;
- (ii) $(U, \mathcal{P})$ is (h, k) -trichotomic;
- (iii) $(U, \mathcal{P})$ is weakly (h, k) -trichotomic.

Proof. By Remark 5.2, we have to prove that $w - (h, k) - t$ implies $s - (h, k) - t$. Let $N \geq 1, a > 0, b \geq 0, c > 0$ given by the $w - (h, k) - t$ property and let $M \geq 1, d \geq 0$ be such that

$$\|P_i(t)\| \leq Mk(t)^d,$$

for every $t \geq 0$ and $i \in \{1, 2, 3\}$.

The conclusion follows from the bellow estimations:

$$\begin{aligned} h(t)^a \|U(t, s)P_1(s)\| &\leq Nh(s)^a k(s)^b \|P_1(s)\| \leq MNh(s)^a k(s)^{b+d}, \\ h(t)^a \|V(t, s)P_2(t)\| &\leq Nh(s)^a k(t)^b \|P_2(t)\| \leq MNh(s)^a k(t)^{b+d}, \\ h(s)^c \|U(t, s)P_3(s)\| &\leq Nh(t)^c k(s)^b \|P_3(s)\| \leq MNh(t)^c k(s)^{b+d}, \\ h(s)^c \|V(t, s)P_3(t)\| &\leq Nh(t)^c k(t)^b \|P_3(t)\| \leq MNh(t)^c k(t)^{b+d}, \end{aligned}$$

for all $t \geq s \geq 0$.

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