

Discrete Bernstein type operators which preserve e_0 and $e_j, j \geq 1$

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ABSTRACT. Starting with the operator introduced in [3] and using quadratures with positive coefficients and a certain degree of exactness we construct positive linear operators of discrete type which preserve e_0 and $e_j, j \geq 1$. We give conditions on the degree of exactness of the quadratures such that the new operators preserve properties of the original operator, such as the convergence, the degree of approximation, the asymptotic formula.

KEY WORDS: positive linear operators, quadratures, convergence, approximation error, asymptotic formula

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1 Introduction and preliminaries

Let $n, j \in \mathbb{N}$, $n \geq j + 1$. In [3] we constructed the following integral operator

$$(1.1) \quad D_{n,j}f(x) = f(0) \sum_{k=0}^{j-1} p_{n,k}(x) + f(1)p_{n,k}(x)$$

$$(1.2) \quad + (n-j) \sum_{k=j}^{n-1} p_{n,k}(x) \int_0^1 p_{n-j-1,k-j}(t) f(t) dt, \quad x \in [0, 1],$$

where

$$p_{n,k}(x) = \binom{n}{k} x^k (1-x)^{n-k}$$

and f is an integrable function on $[0, 1]$.

The operator $D_{n,j}$ preserves the function e_0 and e_j , where $e_i(x) = x^i$. Other operators with the same property were presented in [1], [5], [6] (see also [7], [4] for the operators which preserve a function τ with certain properties).

The following properties of the operator $D_{n,j}$ are proved in [3].

For all $f \in C[0, 1]$ the sequence $(D_{n,j})_{n \geq j+1}$ converges to the function f . We have the following estimation:

$$|D_{n,j}f(x) - f(x)| \leq \left(1 + \frac{1}{\delta^2} \left(\frac{(1-j)(2-j)(1-x)^2 + 2nx(1-x)}{(n-j+1)(n-j+2)} \right)\right) \omega(f, \delta),$$

where $x \in [0, 1]$, $\delta > 0$ and $\omega(f, \cdot)$ is the modulus of continuity, i.e.,

$$\omega(f, \delta) = \sup\{|f(x+h) - f(h)| : x, x+h \in [0, 1], 0 \leq h \leq \delta\}.$$

If f has a second derivative at a point $x \in (0, 1)$, then we have the asymptotic formula

$$\lim_{n \rightarrow \infty} n(D_{n,j}(f)(x) - f(x)) = (1-j)(1-x)f'(x) + x(1-x)f''(x).$$

If $f \in C^2[0, 1]$, then the limit is uniform on $[a, b] \subset (0, 1)$.

We observe that the operator $D_{n,j}$ given by (1.1) uses some integrals as information about the approximated function. In practice it is hard to obtain these integrals. It is known that the Riemann integrals can be approximated by quadrature formulas. In [2] we constructed discrete positive linear operators associated with Durrmeyer operator using quadrature formulas with positive coefficients. In this paper we use the same technique. In order to get discrete operators which fix the function e_0 and e_j we use quadrature formulas with degree of exactness at least j . Such quadratures are Gauss and Fejer-Clenshaw-Curtis quadratures. We study the uniform convergence of these operators. We give an estimation for the approximation error and a Voronovskaya theorem. As the discrete operators preserve the moments of the original operator up to a given order we get the same results for the new operators as in the case of the operator $D_{n,j}$. Some examples are given in the last section.

2 Main result

Using the quadratures

$$(2.1) \quad (n-j) \int_0^1 p_{n-j-1,k-j}(t)f(t)dt \approx \sum_{l=0}^m A_{l,m}^{n,k,j} f(t_{l,m}^{n,k,j}), \quad k = j, \dots, n-1,$$

with positive coefficients, i.e. $A_{l,m}^{n,k,j} \geq 0$, $l = 0, \dots, m$, $k = j, \dots, n-1$, we get the associated discrete operator

$$\begin{aligned} D_{n,j,m}f(x) &= f(0) \sum_{k=0}^{j-1} p_{n,k}(x) + f(1)p_{n,k}(x) \\ &+ \sum_{k=j}^{n-1} p_{n,k}(x) \sum_{l=0}^m A_{l,m}^{n,k,j} f(t_{l,m}^{n,k,j}), \quad x \in [0, 1], \quad f \in C[0, 1]. \end{aligned}$$

The operator $D_{n,j,m}$ uses the values of the function f in the nodes $t_{l,m}^{n,k,j}$ as information about the approximated function. Discrete operators associated with some positive operators using a probabilistic approach were obtained in [9].

The remainder terms of the quadratures (2.1) are given by

$$R_{n,j,m}^k(f) = (n-j) \int_0^1 p_{n-j-1,k-j}(t) f(t) dt - \sum_{l=0}^m A_{l,m}^{n,k,j} f(t_{l,m}^{n,k,j}), \quad k = j, \dots, n-1.$$

If the quadrature formulas have the degree of exactness r , i.e.

$$\begin{aligned} R_{n,j,m}^k(e_i) &= 0, \quad i = 0, \dots, r \\ R_{n,j,m}^k(e_{r+1}) &\neq 0 \end{aligned}$$

then the associated discrete operator has the same images of the monomials e_i , $i = 0, \dots, r$ as the original operator, i.e.

$$D_{n,j,m}(e_i) = D_{n,j}(e_i), \quad i = 0, \dots, r.$$

Theorem 2.1 *If the quadratures (2.1) have the degree of exactness at least two, then the sequence $(D_{n,j,m}(f))_{n \geq j}$ converges uniformly to the function f , for every $f \in C[0, 1]$.*

Proof. We have

$$(2.2) \quad D_{n,j,m}(e_i) = D_{n,j}(e_i), \quad i = 0, 1, 2.$$

Taking into account the uniform convergence of the operator $D_{n,j}$ on the test functions e_i , $i = 0, 1, 2$ and using the well known Korovkin theorem we get the conclusion. ■

Theorem 2.2 *If the quadratures (2.1) have the degree of exactness at least two, then we have the estimation*

$$(2.3) \quad |D_{n,j,m}f(x) - f(x)| \leq \left(1 + \frac{1}{\delta^2} \left(\frac{(1-j)(2-j)(1-x)^2 + 2nx(1-x)}{(n-j+1)(n-j+2)} \right)\right) \omega(f, \delta),$$

for every $f \in C[0, 1]$, where $x \in [0, 1]$ and $\delta > 0$.

The proof follows by applying the Shisha Mond result from [8] and by using (2.2).

Theorem 2.3 *If $f \in C[0, 1]$ has a second derivative at a point $x \in (0, 1)$ the quadratures (2.1) have the degree of exactness at least four, then we have*

$$(2.4) \quad \lim_{n \rightarrow \infty} n(D_{n,j,m}(f)(x) - f(x)) = (1-j)(1-x)f'(x) + x(1-x)f''(x).$$

Moreover, if $f \in C^2[0, 1]$, then the limit is uniform on $[a, b] \subset (0, 1)$.

Proof. We have

$$D_{n,j,m}(e_i) = D_{n,j}(e_i), \quad i = 0, \dots, 4.$$

Following the proof of Theorem 6 from [3] we get the conclusion. ■

3 Examples

In order to obtain the discrete operators we approximate the integrals

$$\int_0^1 t^{k-j}(1-t)^{n-k-1}f(t)dt, \quad k = j, \dots, n-1.$$

Using the substitution $t = (1+u)/2$ we get

$$(3.1) \quad \frac{1}{2^{n-j}} \int_{-1}^1 (1+u)^{k-j}(1-u)^{n-k-1}f\left(\frac{1+u}{2}\right)du, \quad k = j, \dots, n-1.$$

Example 3.1 *To approximate the integrals from (3.1) we can use the Gauss Jacobi quadratures*

$$(3.2) \quad \int_{-1}^1 (1+u)^\alpha(1-u)^\beta g(u)du \approx \sum_{l=0}^m B_{l,m}g(u_{l,m}).$$

Taking

$$\alpha = k-j, \quad \beta = n-k-1, \quad g(u) = f\left(\frac{1+u}{2}\right),$$

the quadrature formulas (3.2) become, for $k=j, \dots, n-1$,

$$\int_{-1}^1 (1+u)^{k-j} (1-u)^{n-k-1} f\left(\frac{1+u}{2}\right) du \approx \sum_{l=0}^m B_{l,m}^{n,k,j} f\left(\frac{1+u_{l,m}^{n,k,j}}{2}\right).$$

The nodes $u_{l,m}^{n,k,j}$, $l = 0, \dots, m$, $k = j, \dots, n-1$ are the roots of the Jacobi orthogonal polynomial of degree $m+1$:

$$J_{m+1}^{(k-j, n-k-1)}(u) = \frac{1}{2^{m+1}(m+1)!} \frac{1}{\rho(u)} \frac{d^{m+1}}{du^{m+1}} [\rho(u)(u^2-1)^{m+1}], \quad u \in (-1, 1),$$

where

$$\rho(u) = (1+u)^{k-j} (1-u)^{n-k-1}.$$

Using [10, Th. 11.5.3], we get the coefficients

$$B_{l,m}^{n,k,j} = \frac{2^{n-j-1} (2m+n-j+1) (m+k-j)! (m+n-k-1)!}{(m+1)! (m+n-j)! J_m^{(k-j, n-k-1)}(u_{l,m}^{n,k,j}) \frac{d}{du} \left[J_{m+1}^{(k-j, n-k-1)}(u) \right]_{u=u_{l,m}^{n,k,j}}}$$

for $l = 0, \dots, m$ and $k = j, \dots, n-1$.

The associated discrete operator is

$$\begin{aligned} D_{n,j,m}^{GJ} f(x) = & f(0) \sum_{k=0}^{j-1} p_{n,k}(x) + f(1) p_{n,k}(x) \\ & + \frac{n-j}{2^{n-j}} \sum_{k=j}^{n-1} p_{n,k}(x) \binom{n-j-1}{k-j} \sum_{j=0}^m B_{l,m}^{n,k,j} f\left(\frac{1+u_{l,m}^{n,k,j}}{2}\right), \end{aligned}$$

where $x \in [0, 1]$ and $f \in C[0, 1]$.

We have

$$D_{n,j,m}^{GJ} e_i = D_{n,j} e_i, \quad i = 0, \dots, 2m+1.$$

By using Theorem 2.1, Theorem 2.2 and Theorem 2.3, it follows that:

- if $m \geq (j-1)/2$, then the operator $D_{n,j,m}^{GJ}$ fixes the function e_j ,
- if $m \geq 1$, then the sequence $(D_{n,j,m}^{GJ} f)_{n \geq 1}$ converges uniformly to the

function f , for every $f \in C[0, 1]$,

- if $m \geq 1$, then we get the estimation of the approximation error for the operator $D_{n,j,m}^{GJ}$ given by (2.3),
- if $m \geq 2$, then we get the asymptotic formula for the operator $D_{n,j,m}^{GJ}$ given by (2.4).

Example 3.2 We take

$$g_{n,k}(u) = (1+u)^{k-j}(1-u)^{n-k-1}f\left(\frac{1+u}{2}\right)$$

and we use the Gauss Legendre quadratures

$$(3.3) \quad \int_{-1}^1 g_{n,k}(u)du \approx \sum_{l=0}^m B_{l,m} g_{n,k}(u_{l,m}), \quad k = j, \dots, n-1.$$

The discrete operator is

$$(3.4) \quad \begin{aligned} D_{n,j,m}^{GL} f(x) = & f(0) \sum_{k=0}^{j-1} p_{n,k}(x) + f(1) p_{n,k}(x) + \frac{n-j}{2^{n-j}} \sum_{k=j}^{n-1} p_{n,k}(x) \binom{n-j-1}{k-j} \\ & \times \sum_{l=0}^m B_{l,m} (1+u_{l,m})^{k-j} (1-u_{l,m})^{n-k-1} f\left(\frac{1+u_{l,m}}{2}\right), \end{aligned}$$

where $x \in [0, 1]$ and $f \in C[0, 1]$. The nodes $u_{l,m}$, $l = 0, \dots, m$ are roots of the Legendre orthogonal polynomial

$$L_{m+1}(u) = \frac{1}{2^{m+1}(m+1)!} \frac{d^{m+1}}{du^{m+1}} \left[(u^2 - 1)^{m+1} \right]$$

and the coefficients are given by (see [10, Th. 11.6.2])

$$B_{l,m} = \frac{2}{(m+1)L_m(u_{l,m}) \frac{d}{du} [L_{m+1}(u)]_{u=u_{l,m}}}, \quad l = 0, \dots, m.$$

We have

$$D_{n,j,m}^{GL} e_i = D_{n,j} e_i, \quad i = 0, \dots, 2m - n + j + 2.$$

It follows that:

- if $m \geq (n - 2)/2$, then the operator $D_{n,j,m}^{GL}$ fixes the function e_j ,
- if $m \geq (n - j)/2$, then the sequence $(D_{n,j,m}^{GL} f)_{n \geq 1}$ converges uniformly to the function f , for every $f \in C[0, 1]$,
- if $m \geq (n - j)/2$, then we get the estimation of the approximation error for the operator $D_{n,j,m}^{GL}$ given by (2.3),
- if $m \geq (n - j)/2 + 1$, then we get the asymptotic formula for the operator $D_{n,j,m}^{GL}$ given by (2.4).

Example 3.3 We approximate the integral from (3.1) using some practical quadratures Fejer-Clenshaw-Curtis. These quadratures are of the form (3.3). From [10, Prop. 9.5.6] we have

$$(3.5) \quad B_{0,m} = B_{m,m} = \begin{cases} \frac{1}{m^2-1}, & m \text{ even} \\ \frac{1}{m^2}, & m \text{ odd} \end{cases}$$

and

$$(3.6) \quad B_{l,m} = B_{m-l,m} = \frac{4}{m \sin \frac{l\pi}{m}} \sum_{j=0}^{\lfloor \frac{m-2}{2} \rfloor} \left(\frac{1}{1-4j^2} + \frac{1}{1-4(j+1)^2} \right) \frac{\sin \frac{(2j+1)l\pi}{m}}{2j+1}, \quad l = \overline{1, m-1}.$$

The nodes are given by (see also [10, p. 108])

$$(3.7) \quad u_{l,m} = \cos \frac{l\pi}{m}, \quad l = \overline{0, m}.$$

The associated discrete operator $D_{n,m}^{FCC}$ has the form (3.4) with $B_{l,m}$ given by (3.5) and (3.6) and $u_{l,m}$ from (3.7).

We have

$$D_{n,j,m}^{FCC} e_i = D_{n,j} e_i, \quad i = 0, \dots, m - n + j + 1.$$

It follows that:

- if $m \geq n - 1$, then the operator $D_{n,j,m}^{FCC}$ fixes the function e_j ,
- if $m \geq n - j + 1$, then the sequence $(D_{n,j,m}^{FCC} f)_{n \geq 1}$ converges uniformly to the function f , for every $f \in C[0, 1]$,
- if $m \geq n - j + 1$, then we get the estimation of the approximation error for the operator $D_{n,j,m}^{FCC}$ given by (2.3),
- if $m \geq n - j + 3$, then we get the asymptotic formula for the operator $D_{n,j,m}^{FCC}$ given by (2.4).

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