

On a nonconvex hyperbolic differential inclusion of third order

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ABSTRACT. We prove the Lipschitz dependence on the initial data of the solution set of a Darboux problem associated to a third order hyperbolic differential inclusion by using the contraction principle in the space of selections of the multifunction instead of the space of solutions. A Filippov type existence theorem for this problem is also provided.

KEY WORDS: set-valued map, hyperbolic differential inclusion, contraction principle

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1 Introduction

This paper is concerned with the following Darboux problem for a third order hyperbolic differential inclusion

(1.1)

$$u_{xyz}(x, y, z) \in F(x, y, z, u(x, y, z)), \quad (x, y, z) \in \Pi := [0, T_1] \times [0, T_2] \times [0, T_3]$$

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with the initial values

$$(1.2) \quad \begin{aligned} u(x, y, 0) &= \varphi(x, y), & (x, y) &\in \Pi_1 := [0, T_1] \times [0, T_2], \\ u(0, y, z) &= \psi(y, z), & (y, z) &\in \Pi_2 := [0, T_2] \times [0, T_3], \\ u(x, 0, z) &= \chi(x, z), & (x, z) &\in \Pi_3 := [0, T_1] \times [0, T_3], \end{aligned}$$

where φ, ψ, χ are absolutely continuous functions satisfying

$$(1.3) \quad \begin{aligned} u(x, 0, 0) &= \varphi(x, 0) = \chi(x, 0) =: v_1(x), & x &\in [0, T_1], \\ u(0, y, 0) &= \varphi(0, y) = \psi(y, 0) =: v_2(y), & y &\in [0, T_2], \\ u(0, 0, z) &= \psi(0, z) = \chi(0, z) =: v_3(z), & z &\in [0, T_3], \\ u(0, 0, 0) &= v_1(0) = v_2(0) = v_3(0) =: v_0 \end{aligned}$$

and $T_1, T_2, T_3 > 0$ and $F : \Pi \times \mathbf{R}^n \rightarrow \mathcal{P}(\mathbf{R}^n)$ is a set-valued map.

Qualitative properties, existence results and structure of the set of solutions of the Darboux problem (1.1)-(1.2) have been studied by many authors [5, 9, 10, 11] etc..

In the present paper we study the properties of the map that associates to given initial conditions the set of solutions of problem (1.1)-(1.2) and the main purpose is to prove that this solution map depends Lipschitz-continuously on the initial conditions. Our approach is based on an idea of Tallos ([6, 8]), applying the set-valued contraction principle in the space of selections of the multifunction instead of the space of solutions as usual (e.g., [11]). This approach allows us to obtain a Filippov type existence result for solutions of problem (1.1)-(1.3). Recall that for a differential inclusion defined by a Lipschitzian set-valued map with nonconvex values, Filippov's theorem consists in proving the existence of a solution starting from a given "quasi" or "almost" solution. Moreover, the result provides an estimate between the "quasi" solution and the solution obtained.

The results in this paper may be interpreted as extensions of similar results obtained for second order hyperbolic differential inclusions to third order hyperbolic differential inclusions ([2, 3]).

The paper is organized as follows: in Section 2 we recall some preliminary results that we use in the sequel and in Section 3 we prove our main results.

2 Preliminaries

Let $\mathcal{P}(\mathbf{R}^n)$ be the family of all nonempty subsets of $\mathbf{R}^n$ and let $\Pi := [0, T_1] \times [0, T_2] \times [0, T_3]$, $\Pi_1 := [0, T_1] \times [0, T_2]$, $\Pi_2 := [0, T_2] \times [0, T_3]$, $\Pi_3 := [0, T_1] \times [0, T_3]$. Given $a \in \mathbf{R}$ we denote by L^1 the Banach space of all (equivalence classes of) Lebesgue measurable functions $\sigma : \Pi \rightarrow \mathbf{R}^n$, endowed with the norm

$$\|\sigma\|_1 = \int_{\Pi} \int_{\Pi} e^{-a(x+y+z)} \|\sigma(x, y, z)\| dx dy dz.$$

We denote by C the Banach space of all continuous functions $u : \Pi \rightarrow \mathbf{R}^n$ endowed with the norm

$$\|u\|_C = \sup_{(x,y,z) \in \Pi} \|u(x, y, z)\|.$$

In what follows by Λ we mean the linear subspace of C consisting of all $\lambda \in C$ such that there exist continuous functions $\varphi : \Pi_1 \rightarrow \mathbf{R}^n$, $\psi : \Pi_2 \rightarrow \mathbf{R}^n$, $\chi : \Pi_3 \rightarrow \mathbf{R}^n$ satisfying (1.3) with $\lambda(x, y, z) = \varphi(x, y) + \psi(y, z) + \chi(x, z) - \varphi(x, 0) - \varphi(0, y) - \psi(0, z) + \psi(0, 0) = \varphi(x, y) + \psi(y, z) + \chi(x, z) - v_1(x) - v_2(y) - v_3(z) + v_0$, $(x, y, z) \in \Pi$. Observe that Λ , equipped with the norm of C , is a separable Banach space.

In order to study problem (1.1)-(1.3) we introduce the following assumption.

Hypothesis 2.1 *Let $F : \Pi \times \mathbf{R}^n \rightarrow \mathcal{P}(\mathbf{R}^n)$ be a set-valued map with nonempty closed values, satisfying the following assumptions:*

- i) *The set-valued maps $(x, y, z) \rightarrow F(x, y, z, u)$ is measurable for all $u \in \mathbf{R}^n$.*
- ii) *There exist $k > 0$ such that, for almost all $(x, y, z) \in \Pi$ $F(x, y, z, \cdot)$ is k -Lipschitz, i.e.*

$$d_H(F(x, y, z, u), F(x, y, z, u')) \leq k \|u - u'\| \quad \forall u, u' \in \mathbf{R}^n.$$

where d_H is the Hausdorff-Pompeiu generalized metric on $\mathcal{P}(\mathbf{R}^n)$ defined by

$$d_H(A, B) = \max\{d^*(A, B), d^*(B, A)\}, \quad d^*(A, B) = \sup\{d(a, B); a \in A\}.$$

iii) $d(0, F(x, y, 0)) \leq k$ for almost all $(x, y, z) \in \Pi$.

For $\sigma \in L^1$, consider the following Darboux problem:

$$(2.1) \quad \begin{aligned} u_{xyz}(x, y, z) &= \sigma(x, y, z), \\ u(x, y, 0) &= \varphi(x, y), \\ u(0, y, z) &= \psi(y, z), \\ u(x, 0, z) &= \chi(x, z). \end{aligned}$$

Definition 2.1 Let $\lambda \in \Lambda$. The function $u \in C$ given by

$$u(x, y, z) = \lambda(x, y, z) + \int_0^x \int_0^y \int_0^z \sigma(\xi, \eta, \mu) d\xi d\eta d\mu \quad (x, y, z) \in \Pi$$

is said to be a solution of (2.1)

Obviously, problem (2.1) has a unique solution, which will be denoted by $u^{\lambda, \sigma}$.

Definition 2.2 Let Hypothesis 2.1 be satisfied and let $\lambda \in \Lambda$. A function $u \in C$ is said to be a solution of problem (1.1)-(1.2) if there exists a function $\sigma \in L^1$ such that

$$\sigma(x, y, z) \in F(x, y, z, u(x, y, z)) \quad \text{a.e. } (\Pi),$$

$$u(x, y, z) = \lambda(x, y, z) + \int_0^x \int_0^y \int_0^z \sigma(\xi, \eta, \mu) d\xi d\eta d\mu \quad (x, y, z) \in \Pi.$$

We denote by $S(\lambda)$ the solution set of (1.1)-(1.2).

In what follows we define

$$W := \{u \in C; \quad \exists(\lambda, \sigma) \in \Lambda \times L^1 \quad \text{such that} \quad u = u^{\lambda, \sigma}\}$$

and

$$\|u(\cdot, \cdot)\|_W = \|u^{\lambda, \sigma}(\cdot, \cdot)\|_W = \|\lambda(\cdot, \cdot)\|_C + \|\sigma(\cdot, \cdot)\|_1$$

It is easy to verify that $\|\cdot\|_W$ is a norm and $(W, \|\cdot\|_W)$ is a Banach space.

Finally we recall some basic results concerning set valued contractions that we shall use in the sequel.

Let (Z, d) be a metric space and consider a set valued map T on Z with nonempty closed values in Z . T is said to be a λ -contraction if there exists $0 < \lambda < 1$ such that:

$$d(T(x), T(y)) \leq \lambda d(x, y) \quad \forall x, y \in Z$$

If Z is complete, then every set valued contraction has a fixed point, i.e. a point $z \in Z$ such that $z \in T(z)$ (see, for instance, [4]).

We denote by $Fix(T)$ the set of all fixed point of the multifunction T . Obviously, $Fix(T)$ is closed. The next result is proved in [7].

Proposition 2.1 *Let Z be a complete metric space and suppose that T_1, T_2 are λ -contractions with closed values in Y . Then*

$$d(Fix(T_1), Fix(T_2)) \leq \frac{1}{1 - \lambda} \sup_{z \in Z} d(T_1(z), T_2(z)).$$

3 The main results

We are ready now to show that the set of solution of (1.1)-(1.2) depends Lipschitz-continuous on the initial conditions.

Theorem 3.1 *Let Hypothesis 2.1 be satisfied and let $a > \sqrt[3]{k}$.*

Then the map $\lambda \rightarrow S(\lambda)$ is Lipschitz-continuous on Λ with nonempty closed values in W .

Proof. Let us consider $\lambda \in \Lambda, \sigma \in L^1$ and define the following set valued maps

$$M_{\lambda, \sigma}(x, y, z) := F(x, y, z, \lambda(x, y, z) + \int_0^x \int_0^y \int_0^z \sigma(\xi, \eta, \mu) d\xi d\eta d\mu),$$

$$T_\lambda(\sigma) := \{\phi(\cdot) \in L^1; \phi(x, y, z) \in M_{\lambda, \sigma}(x, y, z) \text{ a.e. } (\Pi)\}.$$

We shall prove first that $T_\lambda(\sigma)$ is nonempty and closed for every $\sigma \in L^1$.

The fact that the set valued map $M_{\lambda, \sigma}(\cdot, \cdot, \cdot)$ is measurable is well known. For example the map $(x, y, z) \rightarrow F(x, y, z, \lambda(x, y, z) + \int_0^x \int_0^y \int_0^z \sigma(\xi, \eta,$

$\mu)d\xi d\eta d\mu$) can be approximated by step functions and we can apply Theorem III. 40 in [1]. Since the values of F are closed, with the measurable selection theorem (e.g., Theorem III.6 in [1]) we infer that $M_{\lambda,\sigma}(\cdot, \cdot, \cdot)$ admits a measurable selection $\phi(\cdot, \cdot, \cdot)$. According to Hypothesis 2.1, one has

$$\|\phi(x, y, z)\| \leq d(0, F(x, y, z, 0)) + d_H(F(x, y, z, 0), F(x, y, z, \lambda(x, y, z) + \int_0^x \int_0^y \int_0^z \sigma(\xi, \eta, \mu) d\xi d\eta d\mu)) \leq k(1 + \|\lambda\|_C + \int_0^x \int_0^y \int_0^z \|\sigma(\xi, \eta, \mu)\| d\xi d\eta d\mu).$$

Thus, integrating by parts, we obtain

$$\begin{aligned} \int_0^{T_1} \int_0^{T_2} e^{-a(x+y)} \|\phi(x, y)\| dx dy &\leq \int_0^{T_1} \int_0^{T_2} e^{-a(x+y)} k(1 + \|\lambda\|_C + \\ \int_0^x \int_0^y \|\sigma(\xi, \eta)\| d\xi d\eta) dx dy &= \frac{k}{a^2} (1 + \|\lambda\|_C) (1 - e^{-aT_1}) (1 - e^{-aT_2}) + \\ k \int_0^{T_1} \int_0^{T_2} (\int_\xi^{T_1} \int_\eta^{T_2} e^{-a(x+y)} dx dy) \|\sigma(\xi, \eta)\| d\xi d\eta &\leq \frac{k}{a^2} (1 + \|\lambda\|_C) + \frac{k}{a^2} \|\sigma\|_1 \end{aligned}$$

therefore, if $\phi(\cdot, \cdot, \cdot)$ is a measurable selection of $M_{\lambda,\sigma}(\cdot, \cdot, \cdot)$ then $\phi(\cdot, \cdot, \cdot) \in L^1$ and hence $T_\lambda(\sigma)$ is nonempty.

The set $T_\lambda(\sigma)$ is closed. Indeed, if $\phi_n \in T_\lambda(\sigma)$ and $\|\phi_n - \phi\|_1 \rightarrow 0$, then we can pass to a subsequence ϕ_{n_k} such that $\phi_{n_k}(x, y, z) \rightarrow \phi(x, y, z)$ for a.e. $(x, y, z) \in \Pi$ and we find that $\phi \in T_\lambda(\sigma)$.

The next step of the proof will show that $T_\lambda(\cdot)$ is a contraction on L^1 .

Let $\sigma_1, \sigma_2 \in L^1$ be given, $\phi_1 \in T_\lambda(\sigma_1)$ and let $\varepsilon > 0$. Consider the following set valued map

$$G(x, y, z) := M_{\lambda, \sigma_2}(x, y, z) \cap \{\nu \in \mathbf{R}^n; \quad \|\phi_1(x, y, z) - \nu\| \leq k \|\int_0^x \int_0^y \int_0^z (\sigma_1(\xi, \eta, \mu) - \sigma_2(\xi, \eta, \mu)) d\xi d\eta d\mu\| + \varepsilon\}.$$

Since

$$\begin{aligned} d(\phi_1(x, y, z), M_{\lambda, \sigma_2}(x, y, z)) &\leq d_H(F(x, y, z, \lambda(x, y, z) + \\ \int_0^x \int_0^y \int_0^z \sigma_1(\xi, \eta, \mu) d\xi d\eta d\mu), F(x, y, z, \lambda(x, y, z) + \int_0^x \int_0^y \int_0^z \sigma_2(\xi, \eta, \mu) d\xi \\ d\eta d\mu)) &\leq k \|\int_0^x \int_0^y \int_0^z (\sigma_1(\xi, \eta, \mu) - \sigma_2(\xi, \eta, \mu)) d\xi d\eta d\mu\|, \end{aligned}$$

we deduce that $G(\cdot, \cdot, \cdot)$ has nonempty closed values.

Moreover, according to Proposition III.4 in [1], $G(\cdot, \cdot)$ is measurable.

Let $\phi_2(.,.,.)$ be a measurable selection of $G(.,.,.)$. It follows that $\phi_2 \in T_\lambda(\sigma_2)$ and

$$\begin{aligned} \|\phi_1 - \phi_2\|_1 &= \int_0^{T_1} \int_0^{T_2} \int_0^{T_3} e^{-a(x+y+z)} \|\phi_1(x, y, z) - \phi_2(x, y, z)\| dx dy dz \leq \\ &\int_0^{T_1} \int_0^{T_2} \int_0^{T_3} e^{-a(x+y+z)} k \left(\int_0^x \int_0^y \int_0^z \|\sigma_1(\xi, \eta, \mu) - \sigma_2(\xi, \eta, \mu)\| d\xi d\eta d\mu \right) dx dy dz \\ &+ \varepsilon \int_0^{T_1} \int_0^{T_2} \int_0^{T_3} e^{-a(x+y+z)} dx dy dz = k \int_0^{T_1} \int_0^{T_2} \int_0^{T_3} \left(\int_\xi^{T_1} \int_\eta^{T_2} \int_\mu^{T_3} e^{-a(x+y)} e^{-az} dx dy dz \right) \|\sigma_1(\xi, \eta, \mu) - \sigma_2(\xi, \eta, \mu)\| d\xi d\eta d\mu \\ &+ \varepsilon \int_0^{T_1} \int_0^{T_2} \int_0^{T_3} e^{-a(x+y+z)} dx dy dz. \end{aligned}$$

Since ε is arbitrarily, we deduce that

$$d(\phi_1, T_\lambda(\sigma_2)) \leq \frac{k}{a^3} \|\sigma_1 - \sigma_2\|_1.$$

Replacing $\sigma_1(.,.,.)$ by $\sigma_2(.,.,.)$, we obtain

$$d_H(T_\lambda(\sigma_1), T_\lambda(\sigma_2)) \leq \frac{k}{a^3} \|\sigma_1 - \sigma_2\|_1.$$

Hence $T_\lambda(.)$ is a contraction on L^1 .

Consequently, $T_\lambda(.)$ admits a fixed point $\sigma \in L$. We define

$$u(x, y, z) = \lambda(x, y, z) + \int_0^x \int_0^y \int_0^z \sigma(\xi, \eta, \mu) d\xi d\eta d\mu \quad (x, y, z) \in \Pi$$

and we have that $u(.,.,.) \in S(\lambda)$ is solution to (1.1)-(1.2). Since $Fix(T_\lambda(.))$ is closed, we also get that $S(\lambda)$ is a closed set of W .

We prove next the following inequality

$$(3.1) \quad d_H(T_{\lambda_1}(\sigma), T_{\lambda_2}(\sigma)) \leq \frac{k}{a^3} \|\lambda_1 - \lambda_2\|_C \quad \forall \sigma \in L^1, \lambda_1, \lambda_2 \in C$$

Let us consider the set valued map

$$G_1(x, y, z) = M_{\lambda_2, \sigma}(x, y, z) \cap \{\nu \in \mathbf{R}^n; \|\phi(x, y, z) - \nu\| \leq k\|\lambda_1 - \lambda_2\|_C + \varepsilon\},$$

where $\phi(.,.,.)$ is a measurable selection of $M_{\lambda_1, \sigma}(.,.,.)$ and $\varepsilon > 0$.

With the same arguments used for the set valued map $G(.,.,.)$, we deduce that $G_1(.,.,.)$ is measurable with nonempty closed values. Let $\psi(.,.,.)$ be a

measurable selection of $G_1(.,.,.)$.

Since ε is arbitrarily, we deduce that

$$d(\phi, T_{\lambda_2}(\sigma)) \leq \frac{k}{a^3} \|\lambda_1 - \lambda_2\|_C.$$

Replacing $\lambda_2(.)$ by $\lambda_1(.)$ we obtain (3.1).

From (3.1) and Proposition 2.1 we obtain

$$d_H(Fix(T_{\lambda_1}), Fix(T_{\lambda_2})) \leq \frac{k}{a^3 - k} \|\lambda_1 - \lambda_2\|_C.$$

Let $\lambda_1, \lambda_2 \in \Lambda$ and $u \in S(\lambda_1)$. In particular there exists $\sigma_1 \in Fix(T_{\lambda_1})$ and thus, for every $\varepsilon > 0$ there exists $\sigma_2(.,.) \in Fix(T_{\lambda_2})$ such that

$$\|\sigma_1 - \sigma_2\|_1 \leq \frac{k}{a^3 - k} \|\lambda_1 - \lambda_2\|_C + \varepsilon.$$

Put $w(x, y, z) = \lambda_2(x, y, z) + \int_0^x \int_0^y \int_0^z \sigma_2(\xi, \eta, \mu) d\xi d\eta d\mu$; then $w(.,.) \in S(\lambda_2)$ and

$$\begin{aligned} \|u - w\|_W &= \|\lambda_1 - \lambda_2\|_C + \|\sigma_1 - \sigma_2\|_1 \leq \|\lambda_1 - \lambda_2\|_C + \frac{k}{a^3 - k} \|\lambda_1 - \lambda_2\|_C \\ &+ \varepsilon = \frac{a^3}{a^3 - k} \|\lambda_1 - \lambda_2\|_C + \varepsilon. \end{aligned}$$

We infer that

$$d(u, S(\lambda_2)) \leq \frac{a^3}{a^3 - k} \|\lambda_1 - \lambda_2\|_C + \varepsilon$$

for every $\varepsilon > 0$, which means

$$d(u, S(\lambda_2)) \leq \frac{a^3}{a^3 - k} \|\lambda_1 - \lambda_2\|_C$$

and by interchanging λ_1 with λ_2 , if we denote $L := \frac{a^3}{a^3 - k} > 0$ we obtain

$$d(S(\lambda_1), S(\lambda_2)) \leq L \|\lambda_1 - \lambda_2\|_C$$

and the proof is complete.

Using the same idea as in the proof of Theorem 3.1 one may obtain a Filippov type existence result for problem (1.1)-(1.2).

Theorem 3.2 *Let Hypothesis 2.1 be satisfied, $a > \sqrt[3]{k}$, $\lambda_1, \lambda_2 \in \Lambda$ and let $\omega(.,.) \in C$ be such that*

$$d(\omega_{xyz}(x, y, z), F(x, y, z, \omega(x, y, z))) \leq p(x, y, z) \quad a.e. \text{ (II)},$$

where $p(.,.,.) \in L^1(\Pi)$ and $\omega(x, y, 0) = \lambda_2(x, y, 0), \omega(0, y, z) = \lambda_2(0, y, z), \omega(0, y, z) = \lambda_2(x, 0, z)$.

Then for every $\varepsilon > 0$ there exists $u(.,.,.) \in S(\lambda_1)$ such that for every $(x, y, z) \in \Pi$

$$\|u(x, y, z) - \omega(x, y, z)\| \leq \frac{a^3 e^{a(x+y+z)}}{a^3 - k} [\|\lambda_1 - \lambda_2\|_C + \int_0^{T_1} \int_0^{T_2} \int_0^{T_3} e^{-a(x+y+z)} p(x, y, z) dx dy dz] + \varepsilon.$$

Proof. Consider the following set-valued maps

$$\tilde{F}(x, y, z, u) := F(x, y, z, u) + p(x, y, z)B,$$

$$\tilde{M}_{\lambda, \sigma}(x, y, z) := \tilde{F}(x, y, z, \lambda(x, y, z) + \int_0^x \int_0^y \int_0^z \sigma(\xi, \eta, \mu) d\xi d\eta d\mu),$$

$$\tilde{T}_{\lambda_2}(\sigma) := \{\psi \in L^1; \quad \psi(x, y, z) \in \tilde{M}_{\lambda_2, \sigma}(x, y, z) \quad a.e. \text{ (II)}\},$$

where B be the closed unit ball in $\mathbf{R}^n$

Obviously, $\tilde{F}(.,.)$ satisfies Hypothesis 2.1.

As in the proof of Theorem 3.1 we obtain that $\tilde{T}_{\lambda_2}$ is also a $\frac{k}{a^3}$ -contraction on L^1 with closed nonempty values.

We prove next the following estimate:

$$(3.2) \quad d_H(T_{\lambda_1}(\sigma), \tilde{T}_{\lambda_2}(\sigma)) \leq \frac{k}{a^3} \|\lambda_1 - \lambda_2\|_C + \int_0^{T_1} \int_0^{T_2} \int_0^{T_3} e^{-a(x+y+z)} p(x, y, z) dx dy dz.$$

Let $\phi \in T_{\lambda_1}(\sigma), \delta > 0$ and define

$$G_1(x, y, z) = \tilde{M}_{\lambda_2, \sigma}(x, y, z) \cap \{\nu \in \mathbf{R}^n; \quad \|\phi(x, y) - \nu\| \leq k \|\lambda_1 - \lambda_2\|_C + p(x, y, z) + \delta\}$$

With same arguments as for the set valued map $G(.,.,.)$ in the proof of Theorem 3.1, we deduce that $G_1(.,.,.)$ is measurable with nonempty closed values. Let $\psi(.,.,.) \in \tilde{T}_{\lambda_2}(\sigma)$. One has

$$\begin{aligned} \|\phi - \psi\|_1 &\leq \int_0^{T_1} \int_0^{T_2} \int_0^{T_3} e^{-a(x+y+z)} [k\|\lambda_1 - \lambda_2\|_C + p(x, y, z) + \delta] dx dy dz \\ &\leq \frac{k}{a^3} \|\lambda_1 - \lambda_2\|_C + \int_0^{T_1} \int_0^{T_2} \int_0^{T_3} e^{-a(x+y+z)} p(x, y, z) dx dy dz + \delta \int_0^{T_1} \int_0^{T_2} \int_0^{T_3} e^{-a(x+y+z)} dx dy dz \end{aligned}$$

Since δ is arbitrarily, as above, we obtain (3.2).

Applying Proposition 2.1 we obtain

$$d_H(Fix(T_{\lambda_1}), Fix(\tilde{T}_{\lambda_2})) \leq \frac{k}{a^3} \|\lambda_1 - \lambda_2\|_C + \frac{a^3}{a^3-k} \int_0^{T_1} \int_0^{T_2} \int_0^{T_3} e^{-a(x+y+z)} p(x, y, z) dx dy dz.$$

Since $\omega_{xyz}(.,.,.) \in Fix(\tilde{T}_{\lambda_2})$, it follows that there exists $\sigma(.,.,.) \in Fix(T_{\lambda_1})$ such that

$$(3.3) \quad \|\sigma(.,.,.) - \omega_{xyz}(.,.,.)\|_1 \leq \frac{k}{a^3} \|\lambda_1 - \lambda_2\|_C + \frac{a^3}{a^3-k} \int_0^{T_1} \int_0^{T_2} \int_0^{T_3} e^{-a(x+y+z)} p(x, y, z) dx dy dz + \varepsilon e^{-a(T_1+T_2+T_3)}.$$

We define

$$u(x, y, z) = \lambda_1(x, y, z) + \int_0^x \int_0^y \int_0^z \sigma(\xi, \eta, \mu) d\xi d\eta d\mu, \quad (x, y, z) \in \Pi.$$

One has

$$\begin{aligned} \|u(x, y, z) - \omega(x, y, z)\| &\leq \|\lambda_1(x, y, z) - \lambda_2(x, y, z)\| + \int_0^x \int_0^y \int_0^z \|\sigma(\xi, \eta, \mu) - \omega_{xyz}(\xi, \eta, \mu)\| d\xi d\eta d\mu \\ &\leq \|\lambda_1 - \lambda_2\|_C + e^{a(x+y+z)} \int_0^x \int_0^y \int_0^z e^{-a(\xi+\eta+\mu)} \|\sigma(\xi, \eta, \mu) - \omega_{xyz}(\xi, \eta, \mu)\| d\xi d\eta d\mu \\ &\leq \|\lambda_1 - \lambda_2\|_C + e^{a(x+y+z)} \|\sigma - \omega_{xyz}\|_1. \end{aligned}$$

Combining the last inequality with (3.3) we obtain

$$\begin{aligned} \|u(x, y, z) - \omega(x, y, z)\| &\leq \|\lambda_1 - \lambda_2\|_C + \frac{k}{a^3-k} e^{a(x+y+z)} \|\lambda_1 - \lambda_2\|_C + \\ &e^{a(x+y+z)} \frac{a^3}{a^3-k} \int_0^{T_1} \int_0^{T_2} \int_0^{T_3} e^{-a(x+y+z)} p(x, y, z) dx dy dz + e^{a(x+y+z)} \varepsilon \\ &e^{-a(T_1+T_2+T_3)} \leq \frac{a^3 e^{a(x+y+z)}}{a^3-k} (\|\lambda_1 - \lambda_2\|_C + \int_0^{T_1} \int_0^{T_2} \int_0^{T_3} e^{-a(x+y+z)} p(x, y, z) dx dy dz) + \varepsilon \end{aligned}$$

and the proof is complete.

In particular, if $\omega(.,.,.) = 0$, $\lambda_2(.,.,.) = 0$, $p(x, y, z) \equiv k$ we obtain the next consequence of Theorem 3.2.

Corollary 3.1 *Let Hypothesis 2.1 be satisfied, $a > \sqrt[3]{k}$, let $\lambda \in \Lambda$ be given.*

Then for every $\varepsilon > 0$ there exists $u(.,.,.) \in S(\lambda)$ such that for every $(x, y, z) \in \Pi$

$$\|u(x, y, z)\| \leq \frac{a^3 e^{a(x+y+z)}}{a^3 - k} [\|\lambda\|_C + \frac{k(1-e^{-aT_1})(k(1-e^{-aT_2})(k(1-e^{-aT_3}))}{a^3}] + \varepsilon.$$

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