

Kikkawa-Suzuki type theorems for multivalued contractions with respect to a Pompeiu type metric

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ABSTRACT. The purpose of this paper is to discuss some Kikkawa-Suzuki-type fixed point theorems for generalized multivalued H^+ -contractions. A generalization of this results which is an extension of some results given by Reich, is also obtained.

1 Introduction

Suzuki [8] generalized the well-known Banach contraction principle by introducing a new type of mappings. Then Kikkawa and Suzuki [2] gave another generalization which generalized the results of Suzuki and Nadler. Also Moţ and Petruşel [5] gave another generalization concerning the work of Kikkawa and Suzuki.

The results complement and extend some recent results proved by M. Kikkawa, T. Suzuki [2], T. Lazăr [4], T. Suzuki [8], M.A Khamsi and W.A.

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Kirk [1] and A. Petruşel and I.A. Rus [9] using generalized multivalued H^+ -contractions.

Let (X, d) be a metric space and $\mathcal{P}(X)$ be the set of all subsets of X . We denote

$$P(X) := \{Y \in \mathcal{P}(X) \mid Y \neq \emptyset\}$$

$$P_{cl}(X) := \{Y \in \mathcal{P}(X) \mid Y \text{ is closed}\}$$

$$P_{b,cl}(X) := \{Y \in \mathcal{P}(X) \mid Y \text{ is bounded and closed}\}$$

$$P_{cp}(X) := \{Y \in \mathcal{P}(X) \mid Y \text{ is compact}\}$$

By $B(x, r)$ and respectively $\tilde{B}(x, r)$ we will denote the open and respectively the closed ball centered at $x \in X$ with radius $r > 0$.

The following (generalized) functionals are used in the main sections of the paper.

1. The gap functional generated by d :

$$D_d : P(X) \times P(X) \rightarrow \mathbb{R}_+ \cup \{\infty\}, D_d(A, B) = \inf\{d(a, b) \mid a \in A, b \in B\}.$$

2. The diameter functional:

$$\delta : P(X) \times P(X) \rightarrow \mathbb{R}_+ \cup \{\infty\}, \delta(A, B) = \sup\{d(a, b) \mid a \in A, b \in B\}.$$

3. The excess functional:

$$\rho_d : P(X) \times P(X) \rightarrow \mathbb{R}_+ \cup \{\infty\}, \rho_d(A, B) = \sup\{D_d(a, B) \mid a \in A\}.$$

4. The Hausdorff-Pompeiu functional :

$$H_d : P(X) \times P(X) \rightarrow \mathbb{R}_+ \cup \{\infty\}, H_d(A, B) = \max\{\rho_d(A, B), \rho_d(B, A)\}.$$

5. The Pompeiu functional:

$$H_d^+ : P(X) \times P(X) \rightarrow \mathbb{R}_+ \cup \{\infty\}, H_d^+(A, B) := \frac{1}{2}\{\rho_d(A, B) + \rho_d(B, A)\}.$$

6. The Pompeiu generalized functional

$$\begin{aligned} H^\lambda : P(X) \times P(X) &\rightarrow \mathbb{R}_+ \cup \{\infty\}, \\ H_\lambda(A, B) &= (1 - \lambda)\rho(A, B) + \lambda\rho(B, A), \quad \lambda \in [0, 1]. \end{aligned}$$

7. The Hausdorff-Pompeiu generalized functional

$$\begin{aligned} H^\lambda : P(X) \times P(X) &\rightarrow \mathbb{R}_+ \cup \{\infty\}, \\ H^\lambda(A, B) &= \max\{H_\lambda(A, B), H_\lambda(B, A)\}. \end{aligned}$$

We will avoid the subscript $_d$ when we work with just one metric d on X .

Let (X, d) be a metric space. If $T : X \rightarrow P(X)$ is a multivalued operator, then $x \in X$ is called fixed point for T if and only if $x \in T(x)$. We denote by F_T the fixed point set of T and by $(SF)_T$ the set of all strict fixed points of T , i.e., elements $x \in X$ such that $T(x) = \{x\}$.

Concerning the Pompeiu functional H^+ defined above, we have several properties.

Lemma 1.1 ([7]) *The following conclusions take place:*

- a) H^+ is a metric on $P_{b,cl}(X)$;
- b) H^+ is a generalized metric (in the sense that it can take also infinite values) on $P_{cl}(X)$.

Using the Pompeiu type functional H^+ , the following notion was introduced in [7], see also [6].

Definition 1.1 ([7]) *Let (X, d) be a metric space. A multivalued mapping $T : X \rightarrow P_{b,cl}(X)$ is called H^+ -contraction with constant r if:*

1. *there exists a fixed real number r , $0 < r < 1$ such that for every $x, y \in X$*

$$H^+(T(x), T(y)) \leq rd(x, y),$$

2. *for every $x \in X$, $y \in T(x)$ and for every $\epsilon > 0$ there exists z in $T(y)$ such that*

$$d(y, z) \leq H^+(T(x), T(y)) + \epsilon.$$

Concerning the functional H^+ defined above, we have some nice properties.

Lemma 1.2 ([7]) *We have the following relations :*

$$\frac{1}{2}H(A, B) \leq H^+(A, B) \leq H(A, B).$$

Lemma 1.3 ([10]) *Let (X, d) be a metric space, $A, B \in P(X)$ and $q > 1$. Then for each $a \in A$ there exists $b \in B$ such that $d(a, b) \leq q\rho(A, B)$.*

Remark 1.1 *Following Kirk and Shahzad [3], we will replace the second condition of the Definition 1.1:*

(*) *for every $x \in X$, $y \in T(x)$ and for every $\epsilon > 0$ there exists z in $T(y)$ such that*

$$d(y, z) \leq H^+(T(x), T(y)) + \epsilon$$

with the following one:

(**) *for every $x \in X$ and every $y \in T(x)$ we have that*

$$D(y, T(y)) \leq H^+(T(x), T(y)).$$

*Notice that (**) implies (*). Moreover if we consider the following condition*

(***) *for every $\epsilon > 0$, for every $x \in X$, $y \in T(x)$ there exists z in $T(y)$ such that*

$$d(y, z) \leq H^+(T(x), T(y)) + \epsilon,$$

*then it is easy to see that (**) is equivalent with (***). In this last case, we also notice that for each $x \in X$ and every $y \in T(x)$ we have*

$$\rho(T(x), T(y)) \leq H^+(T(x), T(y)).$$

As a consequence, $\rho(T(x), T(y)) \leq \rho(T(y), T(x))$ and so

$$H^+(T(x), T(y)) \leq \rho(T(y), T(x)), \text{ for each } x \in X \text{ and } y \in T(x).$$

Theorem 1.1 *Let (X, d) be a complete metric space and $T : X \rightarrow P_{cl}(X)$ be a multivalued (H^+, α) -contraction. Then $F_T \neq \emptyset$.*

Proof. Let $x_0 \in X$ be arbitrary. Fix an element $x_1 \in T(x_0)$. From the Remark 1 and the definition of (H^+, α) -contraction we get that :

$$\rho(T(x_0), T(x_1)) \leq H^+(T(x_0), T(x_1)) \leq \alpha d(x_0, x_1).$$

From Lemma 1.3 we obtain that for $q \in (1, \frac{1}{\alpha})$ there exists $x_2 \in T(x_1)$ such that

$$d(x_1, x_2) \leq \rho(T(x_0), T(x_1)) \leq q\alpha d(x_0, x_1).$$

In general, if x_n is chosen, then we choose $x_{n+1} \in T(x_n)$ such that

$$d(x_n, x_{n+1}) \leq q\alpha d(x_{n-1}, x_n).$$

Repeating the same argument n-times we get a sequence of successive approximations for T starting from $(x_0, x_1) \in \text{Graph}(T)$ such that, for each $n \in \mathbb{N}$, we have

$$d(x_n, x_{n+1}) \leq (q\alpha)^n d(x_0, x_1).$$

Then,

$$d(x_n, x_{n+p}) \leq (q\alpha)^n \frac{1 - (q\alpha)^p}{1 - q\alpha} d(x_0, x_1), \text{ for each } n \in \mathbb{N}^*, p \in \mathbb{N}^*.$$

This implies that the sequence $(x_n)_{n \in \mathbb{N}}$ is Cauchy and hence convergent in (X, d) to some $x^* \in X$. Notice that, by the contraction condition, we immediately get that $\text{Graph}(T)$ is closed in $X \times X$. Hence $x^* \in F_T$. ■

The first result is an extension of the result proved by Kikkawa-Suzuki [2] using the Pompeiu functional.

Theorem 1.2 *Let (X, d) be a complete metric space, let $T : X \rightarrow P_{cl}(X)$ be a multivalued operator and $\eta : [0, 1) \rightarrow (\frac{1}{3}, 1]$ given by $\eta(t) = \frac{1}{1+2t}$. Assume that there exists $r \in [0, 1)$ such that $\eta(r)D(x, T(x)) \leq d(x, y)$ implies that T is a H^+ -contraction with constant r . Then, $\text{Fix}T \neq \emptyset$.*

Proof.

Take a real number r_1 with $0 \leq r < r_1 < 1$. Then for $u_0 \in X$ and $u_1 \in T(u_0)$, we have

$$\eta(r)D(u_0, T(u_0)) \leq \eta(r)d(u_0, u_1) \leq d(u_0, u_1).$$

From the assumption, we get that

$$H^+(T(u_0), T(u_1)) \leq rd(u_0, u_1).$$

Using the second relation of the definition of H^+ -contraction with constant r we obtain that for every $\epsilon > 0$ there exist $u_2 \in T(u_1)$ such that

$$d(u_1, u_2) \leq H^+(T(u_0), T(u_1)) + \epsilon \leq rd(u_0, u_1) + \epsilon.$$

In general, if u_n is chosen, then we choose $u_{n+1} \in T(u_n)$ such that

$$d(u_n, u_{n+1}) \leq H^+(T(u_{n-1}), T(u_n)) + \epsilon \leq rd(u_{n-1}, u_n) + \epsilon.$$

Set $\epsilon_n := (r_1 - r)d(u_{n-1}, u_n)$. Let $r_1 \in \mathbb{R}$ such that $0 \leq r < r_1 < 1$. Then, from the above relation it follows that

$$d(u_n, u_{n+1}) \leq r_1 d(u_{n-1}, u_n).$$

Then,

$$\sum_{n=1}^{\infty} d(u_n, u_{n+1}) \leq \sum_{n=1}^{\infty} r_1^{n-1} d(u_0, u_1) < \infty$$

and hence $\{u_n\}$ is a Cauchy sequence. Since X is complete, $\{u_n\}$ converges to some point $z \in X$.

We next show that

$$D(z, T(x)) \leq rd(z, x) \text{ for all } x \in X \setminus \{z\}.$$

Since $u_n \rightarrow z$, there exists $\nu \in \mathbb{N}$ such that

$$d(z, u_n) \leq \frac{1}{3}d(z, x) \text{ for all } n \in \mathbb{N} \text{ with } n \geq \nu.$$

Then we have

$$\begin{aligned}
 \eta(r)D(u_n, T(u_n)) &\leq D(u_n, T(u_n)) \leq d(u_n, u_{n+1}) \\
 &\leq d(u_n, z) + d(u_{n+1}, z) \leq \frac{2}{3}d(x, z) \\
 &= d(x, z) - \frac{1}{3}d(x, z) \leq d(x, z) - d(u_n, z) \leq d(u_n, z)
 \end{aligned}$$

and hence $H^+(T(u_n), T(x)) \leq rd(u_n, x)$. Since,

$$H(T(u_n), T(x)) \leq 2H^+(T(u_n), T(x)) \leq 2rd(u_n, x)$$

and $D(u_{n+1}, T(x)) \leq H(T(u_n), T(x))$ we get that

$$D(u_{n+1}, T(x)) \leq 2rd(u_n, x) \text{ for all } n \in \mathbb{N} \text{ with } n \geq \nu.$$

Letting n tend to ∞ , we obtain

$$D(z, T(x)) \leq rd(z, x) \text{ for all } x \in X \setminus \{z\}.$$

Next, we will prove that $H^+(T(x), T(z)) \leq rd(x, z)$ for all $x \in X$. If $x = z$, then the inequality obviously holds. So we assume $x \neq z$. Then, for every $n \in \mathbb{N}$, there exists $y_n \in T(x)$ such that

$$d(z, y_n) \leq D(z, T(x)) + \frac{1}{n}d(x, z).$$

We have

$$\begin{aligned}
 D(x, T(x)) &\leq d(x, y_n) \leq d(x, z) + d(z, y_n) \\
 &\leq d(x, z) + D(z, T(x)) + \frac{1}{n}d(x, z) \\
 &\leq d(x, z) + 2rd(x, z) + \frac{1}{n}d(x, z) \\
 &= (1 + 2r + \frac{1}{n})d(x, z),
 \end{aligned}$$

for $n \in \mathbb{N}$ and hence $\eta(r)D(x, T(x)) \leq d(x, z)$. From the assumption, we

have $H^+(T(x), T(z)) \leq rd(x, z)$. Finally, since

$$\begin{aligned} d(z, T(z)) &= \lim_{n \rightarrow \infty} D(u_{n+1}, T(z)) \leq \lim_{n \rightarrow \infty} H(T(u_n), T(z)) \\ &\leq \lim_{n \rightarrow \infty} 2H^+(T(u_n), T(z)) \leq \lim_{n \rightarrow \infty} 2rd(u_n, z) = 0 \end{aligned}$$

and since $T(z)$ is closed, we obtain $z \in T(z)$. ■

Following Reich [11], our next result extend the Kikkawa-Suzuki's theorem. We introduced the following concept.

Definition 1.2 Let (X, d) be a metric space. Then, $T : X \rightarrow P_{cl}(X)$ is called an (a, b, c) -KSR (Kikkawa-Suzuki-Reich) multivalued operator if $a, b, c \in \mathbb{R}_+$ with $a + b + c \in (0, 1)$ we have $\frac{1 - 2b - 2c}{1 + 2a}D(x, T(x)) \leq d(x, y)$ implies that

- (i) $H^+(T(x), T(y)) \leq ad(x, y) + bD(x, T(x)) + cD(y, T(y))$, for $x, y \in X$
- (ii) for every $x \in X$, $y \in T(x)$ and $\epsilon > 0$ there exists $z \in T(y)$ such that

$$d(y, z) \leq H^+(T(x), T(y)) + \epsilon.$$

Theorem 1.3 Let (X, d) be a complete metric space and $T : X \rightarrow P_{cl}(X)$ be a (a, b, c) -KSR multivalued operator. Then $\text{Fix}(T) \neq \emptyset$.

Proof.

Take a real number r_1 with $0 \leq r < r_1 < 1$. Then, for each $u_0 \in X$ and $u_1 \in T(u_0)$, we have

$$D(u_0, T(u_0)) \leq \eta d(u_0, u_1) \leq d(u_0, u_1).$$

From the assumption, we get that

$$H^+(T(u_0), T(u_1)) \leq ad(u_0, u_1) + bD(u_0, T(u_0)) + cD(u_1, T(u_1)).$$

From the definition of (a, b, c) -KSR (ii), we obtain that for $u_0 \in X$,

$u_1 \in T(u_0)$ and $\epsilon > 0$ there exists $u_2 \in T(u_1)$ such that

$$\begin{aligned} d(u_1, u_2) &\leq H^+(T(u_0), T(u_1)) + \epsilon \\ &\leq ad(u_0, u_1) + bD(u_0, T(u_0)) + cD(u_1, T(u_1)) + \epsilon \\ &\leq ad(u_0, u_1) + bd(u_0, u_1) + cd(u_1, u_2) + \epsilon \\ &\leq (a+b)d(u_0, u_1) + cd(u_1, u_2) + \epsilon. \end{aligned}$$

In general, if u_n is chosen, then we choose $u_{n+1} \in T(u_n)$ such that

$$\begin{aligned} d(u_n, u_{n+1}) &\leq H^+(T(u_{n-1}), T(u_n)) + \epsilon \\ &\leq (a+b)d(u_{n-1}, u_n) + cd(u_n, u_{n+1}) + \epsilon. \end{aligned}$$

By setting $\epsilon_n := qd(u_n, u_{n+1})$ with $q < 1 - (a+b+c)$ we get that

$$d(u_n, u_{n+1}) \leq \frac{a+b}{1-c-q} d(u_{n-1}, u_n).$$

In a similar way, we will obtain a sequence $(x_n)_{n \in \mathbb{N}}$ of successive approximation for T starting from x_0 , satisfying the following assertions :

- (a) $x_{n+1} \in T(x_n)$, for each $n \in \mathbb{N}$;
- (b) $d(u_n, u_{n+1}) \leq (\frac{a+b}{1-c-q})^n d(x_0, x_1)$, for $n \in \mathbb{N}$.

Denote $p := \frac{a+b}{1-c-q} < 1$. From (b) we get that

$$d(u_n, u_{n+p}) \leq \frac{p^n}{1-p} d(x_0, x_1), \text{ for each } n \in \mathbb{N} \text{ and } p \in \mathbb{N}^*.$$

Letting $n \rightarrow +\infty$, we obtain that the sequence $(x_n)_{n \in \mathbb{N}}$ is Cauchy in (X, d) . Denote by z its limit. We will prove that z is a fixed point for T . We will show first that

$$D(z, T(x)) \leq \left(a + \frac{b}{\eta}\right) d(z, x) + cD(x, T(x)), \text{ for each } x \in X \setminus \{z\}.$$

Suppose $x \neq z$. Since $x_n \rightarrow z$ as $n \rightarrow +\infty$, there exists $n_0 \in \mathbb{N}^*$ such that

$d(x_n, z) \leq \frac{1}{3}d(z, x)$ for each $n \leq n_0$. Then we have :

$$\begin{aligned} \eta D(x_n, T(x_n)) &\leq D(x_n, T(x_n)) \leq d(x_n, x_{n+1}) \\ &\leq d(x_n, z) + d(z, x_{n+1}) \leq \frac{2}{3}d(x, z) \\ &\leq d(x, z) - d(x_n, z) \leq d(x_n, x). \end{aligned}$$

Hence $H^+(T(x_n), T(x)) \leq \left(a + \frac{b}{\eta}\right)d(x_n, x) + cD(x, T(x))$, for each $n \leq n_0$. Thus,

$$\begin{aligned} D(x_{n+1}, T(x)) &\leq H(T(x_n), T(x)) \leq 2H^+(T(x_n), T(x)) \\ &\leq 2\left(a + \frac{b}{\eta}\right)d(x_n, x) + 2cD(x, T(x)), \text{ for each } n \leq n_0. \end{aligned}$$

Letting $n \rightarrow +\infty$ we get that

$$D(z, T(x)) \leq 2\left(a + \frac{b}{\eta}\right)d(z, x) + 2cD(x, T(x)), \text{ for all } x \neq z.$$

Next we will show that $H^+(T(x), T(z)) \leq \left(a + \frac{b}{\eta}\right)d(x, z)$, for all $x \in X$.

Again, if $x := z$ then the conclusion obviously holds.

Suppose $x \neq z$. Then for every $n \in \mathbb{N}^*$ there exists $y_n \in T(x)$ such that $d(z, y_n) \leq D(z, T(x)) + \frac{1}{n}d(x, z)$. Then we have :

$$\begin{aligned} D(x, T(x)) &\leq d(x, y_n) \leq d(x, z) + d(z, y_n) \\ &\leq d(x, z) + D(z, T(x)) + \frac{1}{n}d(x, z) \\ &\leq d(x, z) + 2\left(a + \frac{b}{\eta}\right)d(x, z) + d(x, z) + 2cD(x, T(x)) + \frac{1}{n}d(x, z) \\ &= \left(1 + 2a + \frac{2b}{\eta} + \frac{1}{n}\right)d(x, z) + 2cD(x, T(x)), \end{aligned}$$

for all $n \in \mathbb{N}^*$.

Hence, as $n \rightarrow +\infty$ we get that

$$(1 - 2c)D(x, T(x)) \leq \left(1 + 2a + \frac{2b}{\eta}\right)d(x, z)$$

and so $\eta D(x, T(x)) \leq d(x, z)$. Thus,

$$\begin{aligned} H^+(T(x), T(z)) &\leq aD(x, z) + bD(x, T(x)) + cD(z, T(z)) \\ &\leq \left(a + \frac{b}{\eta}\right)d(x, z), \text{ for all } x \in X. \end{aligned}$$

Finally, we obtain

$$\begin{aligned} D(z, T(z)) &= \lim_{n \rightarrow +\infty} D(x_{n+1}, T(z)) \leq H(T(x_n), T(z)) \\ &\leq \lim_{n \rightarrow +\infty} 2H^+(T(x_n), T(z)) \leq \lim_{n \rightarrow +\infty} 2\left(a + \frac{b}{\eta}\right)d(x_n, z) = 0. \end{aligned}$$

Thus $z \in \text{Fix}T$. ■

Definition 1.3 Let (X, d) be a metric space. Then, $T : X \rightarrow P_{cl}(X)$ is called an α KSR multivalued operator if $\alpha \leq \frac{1}{2}$ we have $(1 - 2\alpha)D(x, T(x)) \leq d(x, y)$ implies that

$$(i) \quad H^+(T(x), T(y)) \leq \alpha\{d(x, y), D(x, T(x)), D(y, T(y)), \frac{1}{2}(D(x, T(y)) + D(y, T(x)))\}, \text{ for } x, y \in X;$$

(ii) for every $x \in X$, $y \in T(x)$ and for every $q > 1$ there exists $z \in T(y)$ such that

$$d(y, z) \leq qH^+(T(x), T(y)).$$

Theorem 1.4 Let (X, d) be a complete metric space and $T : X \rightarrow P_{cl}(X)$ be a α -KSR multivalued operator. Then $\text{Fix}(T) \neq \emptyset$.

Proof.

Take a real number $\alpha \in (0, \frac{1}{2})$ with $1 < q < \frac{1}{\alpha}$. Then, for each $u_0 \in X$ and $u_1 \in T(u_0)$, we have

$$(1 - 2\alpha)D(u_0, T(u_0)) \leq (1 - 2\alpha)d(u_0, u_1) \leq d(u_0, u_1).$$

From the assumption, we get that

$$\begin{aligned} H^+(T(u_0), T(u_1)) &\leq q\alpha \max\{d(u_0, u_1), D(u_0, T(u_0)), \\ &D(u_1, T(u_1)), \frac{1}{2}(D(u_0, T(u_1)) + D(u_1, T(u_0)))\}. \end{aligned}$$

From the definition of $\alpha - KSR$ (ii), we obtain that for $u_0 \in X$, $u_1 \in T(u_0)$ and $\epsilon > 0$ there exists $u_2 \in T(u_1)$ such that:

$$\begin{aligned} d(u_1, u_2) &\leq qH^+(T(u_0), T(u_1)) \\ &\leq q\alpha \max\{d(u_0, u_1), D(u_0, T(u_0)), \\ &D(u_1, T(u_1)), \frac{1}{2}(D(u_0, T(u_1)) + D(u_1, T(u_0)))\} \\ &\leq q\alpha \max\{d(u_0, u_1), d(u_0, u_1), d(u_1, u_2), \frac{1}{2}d(u_0, u_2)\} \\ &\leq q\alpha \max\{d(u_0, u_1), d(u_1, u_2), \frac{1}{2}(d(u_0, u_1) + d(u_1, u_2))\} \\ &\leq q\alpha \max\{d(u_0, u_1), d(u_1, u_2)\}. \end{aligned}$$

If $d(u_0, u_1) \leq d(u_1, u_2)$ then $d(u_1, u_2) \leq q\alpha d(u_1, u_2) \Leftrightarrow 1 < q\alpha$ which is a contradiction.

Then $d(u_0, u_1) > d(u_1, u_2)$ and we obtain

$$d(u_1, u_2) \leq q\alpha d(u_0, u_1).$$

In general, if u_n is chosen, then we choose $u_{n+1} \in T(u_n)$ such that

$$d(u_n, u_{n+1}) \leq qH^+(T(u_{n-1}), T(u_n)) \leq q\alpha d(u_{n-1}, u_n).$$

In a similar way, we will obtain a sequence $(x_n)_{n \in \mathbb{N}}$ of successive

approximation for T starting from u_0 , satisfying the following assertions:

- (a) $x_{n+1} \in T(x_n)$, for each $n \in \mathbb{N}$;
- (b) $d(u_n, u_{n+1}) \leq (q\alpha)^n d(x_0, x_1)$, for $n \in \mathbb{N}$.

Letting $n \rightarrow +\infty$, we obtain that the sequence $(x_n)_n \in \mathbb{N}$ is Cauchy in

(X, d) . Denote by z its limit. We will prove that z is a fixed point for T . Will show first that

$$D(z, T(x)) \leq 2\alpha \max\left\{\frac{1}{1-2\alpha}d(z, x), D(x, T(x))\right\}, \text{ for each } x \in X \setminus \{z\}.$$

Suppose $x \neq z$. Since $x_n \rightarrow z$ as $n \rightarrow \infty$, there exists $n_0 \in \mathbb{N}^*$ such that $d(x_n, z) \leq \frac{1}{3}d(z, x)$ for each $n \leq n_0$.

Then we have :

$$\begin{aligned} (1-2\alpha)D(x_n, T(x_n)) &\leq D(x_n, T(x_n)) \leq d(x_n, x_{n+1}) \\ &\leq d(x_n, z) + d(z, x_{n+1}) \leq \frac{2}{3}d(x, z) \\ &\leq d(x, z) - d(x_n, z) \leq d(x_n, x). \end{aligned}$$

Hence

$$\begin{aligned} H^+(T(x_n), T(x)) &\leq \alpha \max\{d(x_n, x), D(x_n, T(x_n)), \\ &D(x, T(x)), \frac{1}{2}(D(x_n, T(x)) + D(x, T(x_n)))\} \\ &\leq \alpha \max\{d(x_n, x), \frac{1}{1-2\alpha}d(x_n, x), \\ &D(x, T(x)), \frac{1}{2}(D(x_n, T(x)) + d(x, x_{n+1}))\}, \end{aligned}$$

Thus,

$$\begin{aligned} D(x_{n+1}, T(x)) &\leq H(T(x_n), T(x)) \leq 2H^+(T(x_n), T(x)) \\ &\leq 2\alpha \max\left\{\frac{1}{1-2\alpha}d(x_n, x), D(x, T(x)), \frac{1}{2}(D(x_n, T(x)) + d(x, x_{n+1}))\right\}, \end{aligned}$$

for each $n \leq n_0$.

Letting $n \rightarrow \infty$ we get that

$$D(z, T(x)) \leq 2\alpha \max\left\{\frac{1}{1-2\alpha}d(z, x), D(x, T(x)), \frac{1}{2}(D(z, T(x)) + d(x, z))\right\},$$

for all $x \neq z$.

If $d(x, z) \leq D(z, T(x))$, then

$$\begin{aligned} D(z, T(x)) &\leq 2\alpha \max\left\{\frac{1}{1-2\alpha}d(z, x), D(x, T(x)), D(z, T(x))\right\} \\ &\leq 2\alpha \max\left\{\frac{1}{1-2\alpha}d(z, x), D(x, T(x))\right\} \end{aligned}$$

and if $D(z, T(x)) < d(x, z)$, then

$$D(z, T(x)) \leq 2\alpha \max\left\{\frac{1}{1-2\alpha}d(z, x), D(x, T(x))\right\}.$$

Thus, for any $x \in X \setminus \{z\}$,

$$D(z, T(x)) \leq 2\alpha \max\left\{\frac{1}{1-2\alpha}d(z, x), D(x, T(x))\right\}.$$

Next, we will show that

$$H^+(T(x), T(z)) \leq \alpha \max\left\{\frac{1}{1-2\alpha}d(x, z), D(z, T(z))\right\}.$$

Again, if $x := z$ then the conclusion obviously holds. Suppose $x \neq z$. Then for every $n \in \mathbb{N}^*$ there exists $y_n \in T(x)$ such that

$$d(z, y_n) \leq D(z, T(x)) + \frac{1}{n}d(x, z).$$

Then, we have :

$$\begin{aligned}
 D(x, T(x)) &\leq d(x, y_n) \leq d(x, z) + d(z, y_n) \\
 &\leq d(x, z) + D(z, T(x)) + \frac{1}{n}d(x, z) \\
 &\leq d(x, z) + 2\alpha \max\left\{\frac{1}{1-2\alpha}d(z, x), D(x, T(x))\right\} + \frac{1}{n}d(x, z)
 \end{aligned}$$

If $D(x, T(x)) \leq d(z, x)$, then

$$D(x, T(x)) \leq \left(1 + \frac{2\alpha}{1-2\alpha} + \frac{1}{n}\right)d(z, x).$$

When $n \rightarrow \infty$, we have

$$D(x, T(x)) \leq \left(1 + \frac{2\alpha}{1-2\alpha}\right)d(z, x).$$

Hence, we find that $(1 - 2\alpha)D(x, T(x)) \leq d(z, x)$.

If $d(z, x) < D(x, T(x))$, then

$$D(x, T(x)) \leq d(x, z) + \frac{2\alpha}{1-2\alpha}D(x, T(x)) + \frac{1}{n}d(x, z).$$

Thus, $\left(1 - \frac{2\alpha}{1-2\alpha}\right)D(x, T(x)) \leq \left(1 + \frac{1}{n}\right)d(x, z)$ gives when $n \rightarrow \infty$

$$D(x, T(x)) \leq \frac{1}{1-2\alpha}d(x, z).$$

Now, for every $x \in X \setminus \{z\}$, we have

$$(1 - 2\alpha)D(x, T(x)) \leq d(x, z).$$

Thus,

$$\begin{aligned}
H^+(T(x), T(y)) &\leq \alpha \max\{d(x, z), D(x, T(x)), \\
D(z, T(z)), \frac{1}{2}(D(x, T(z)) + D(z, T(x)))\} \\
&\leq \alpha \max\{d(x, z), \frac{1}{1-2\alpha}d(x, z), D(z, T(z)), \frac{1}{2}(D(x, T(z)) + \frac{2\alpha}{1-2\alpha}d(z, x))\} \\
&\leq \alpha \max\{\frac{1}{1-2\alpha}d(x, z), D(z, T(z)), \frac{1}{2}(d(x, z) + D(z, T(z)) + \frac{2\alpha}{1-2\alpha}d(z, x))\} \\
&\leq \alpha \max\{\frac{1}{1-2\alpha}d(x, z), D(z, T(z)), \frac{1}{2}(\frac{1}{1-2\alpha}d(x, z) + D(z, T(z)))\} \\
&= \alpha \max\{\frac{1}{1-2\alpha}d(x, z), D(z, T(z))\}.
\end{aligned}$$

If $d(x, z) \leq D(z, T(z))$, then we have $H^+(T(x), T(z)) \leq \alpha D(z, T(z))$ and we obtain

$$\begin{aligned}
D(z, T(z)) &= \lim_{n \rightarrow +\infty} D(x_{n+1}, T(z)) \leq H(T(x_n), T(z)) \\
&\leq \lim_{n \rightarrow +\infty} 2H^+(T(x_n), T(z)) \leq \lim_{n \rightarrow +\infty} 2\alpha D(x, T(z)).
\end{aligned}$$

Which is equivalent with $(1-\alpha)D(z, T(z)) \leq 0$, so $D(z, T(z)) = 0$. Thus $z \in \text{Fix}T$.

If $D(z, T(z)) < d(x, z)$, then $H^+(T(x), T(z)) \leq \frac{\alpha}{1-2\alpha}$ and we obtain

$$\begin{aligned}
D(z, T(z)) &= \lim_{n \rightarrow +\infty} D(x_{n+1}, T(z)) \leq H(T(x_n), T(z)) \\
&\leq \lim_{n \rightarrow +\infty} 2H^+(T(x_n), T(z)) \leq \lim_{n \rightarrow +\infty} \frac{2\alpha}{1-2\alpha}d(x_n, z) = 0.
\end{aligned}$$

Thus $z \in \text{Fix}T$.

■

Concerning the Pompeiu functional H^λ defined above, we have several properties.

Lemma 1.4 For $\lambda = \frac{1}{2}$ we obtain that $H^{\frac{1}{2}}(A, B) = H^+(A, B)$. For $\lambda = 1$

or $\lambda = 0$ we get that $H^0(A, B) = H(A, B)$ and $H^1(A, B) = H(A, B)$.

Lemma 1.5 *We have the following relation:*

$$H^+(A, B) \leq H^\lambda(A, B) \leq H(a, B).$$

Proof. We know that the arithmetic mean is greater than the largest element, so

$$H^\lambda(A, B) \geq \frac{H_\lambda(A, B) + H_\lambda(B, A)}{2} = H^+(A, B).$$

To proof the other inequality we consider that $\max\{\rho(A, B), \rho(B, A)\} = \rho(B, A)$, hence $\rho(A, B) \leq \rho(B, A)$. Then,

$$\begin{aligned} H^\lambda(A, B) &= \max\{H_\lambda(A, B), H_\lambda(B, A)\} \\ &= \max\{(1 - \lambda)\rho(A, B) + \lambda\rho(B, A), (1 - \lambda)\rho(B, A) + \lambda\rho(A, B)\} \\ &\leq \rho(A, B) = H(A, B). \end{aligned}$$

■

Lemma 1.6 *Let $A, B \in P_{b,cl}(X)$. If for every $a \in A$ there exists $b \in B$ such that*

$$d(a, b) \leq r_1$$

and for every $b \in B$ there exists $a \in A$ such that

$$d(a, b) \leq r_2.$$

Then

$$H^\lambda(A, B) \leq \max\{(1 - \lambda)r_1 + \lambda r_2, (1 - \lambda)r_2 + \lambda r_1\}.$$

Proof.

From the relations (1.1) and (1.2) we get that $\rho(A, B) \leq r_1$ and $\rho(B, A) \leq r_2$. Multiplying the both relations with $1 - \lambda$ respectively λ

and adding them, we get that

$$(1 - \lambda)\rho(A, B) + \lambda\rho(B, A) \leq (1 - \lambda)r_1 + \lambda r_2.$$

Hence $H_\lambda(A, B) \leq (1 - \lambda)r_1 + \lambda r_2$.

Proceeding in the same way, we obtain : $H_\lambda(B, A) \leq (1 - \lambda)r_2 + \lambda r_1$.
By taking maximum between the above relations, we obtain

$$H^\lambda(A, B) \leq \max\{(1 - \lambda)r_1 + \lambda r_2, (1 - \lambda)r_2 + \lambda r_1\}.$$

■

Lemma 1.7 *If $H^\lambda(A, B) < r$ then there exist $r_1, r_2 > 0$ such that $r = (1 - \lambda)r_1 + \lambda r_2$ and $A \subset V(B, r_1)$, $B \subset V(A, r_2)$.*

Proof.

Let $H^\lambda(A, B) = k < r$. then there exist $k_1, k_2 > 0$ such that $k = (1 - \lambda)k_1 + \lambda k_2$ and

$$\rho(A, B) = k_1, \quad \rho(B, A) = k_2.$$

As $0 < k < r$, it follows that there exist $r_1, r_2 > 0$ such that $k_1 < r_1$, $k_2 < r_2$ and $k = (1 - \lambda)r_1 + \lambda r_2$. Then from the above inequalities it follows that

$$D(x, B) \leq k_1 < r_1, \text{ for every } x \in A$$

and

$$D(A, y) \leq k_2 < r_2, \text{ for every } y \in B.$$

Then, for any $x \in A$ there exist $y_x \in B$ such that

$$d(x, y_x) < D(x, B) + r_1 - k_1 \leq r_1$$

and, for any $y \in B$ there exist $x_y \in A$ such that

$$d(x_y, y) < D(A, y) + r_2 - k_2 \leq r_2.$$

Hence, for any $x \in A$ and $y \in B$ it follows that $x \in \bigcup_{y \in B} V(y, r_1)$ and $y \in \bigcup_{x \in A} V(x, r_2)$, that is $A \subset V(B, r_1)$ and $B \subset V(A, r_2)$. ■

Lemma 1.8 *Let $A, B \in P_{b,cl}(X)$ such that $\rho(A, B) \leq \rho(B, A)$ and $q > 1$. Then for every $b \in B$ there exists $a \in A$ such that $d(a, b) \leq qH^\lambda(A, B)$.*

Proof.

Supposing again contrary: there exists $q > 1$ and there exists $b \in B$ such that for all $a \in A$ we have

$$d(a, b) > qH^\lambda(A, B).$$

Then we have

$$\rho(A, B) \geq D(b, A) \geq qH^\lambda(A, B).$$

From Lemma 2.1 we get that

$$H^\lambda(A, B) \geq \rho(A, B) \geq qH^\lambda(A, B).$$

Hence $q \leq 1$, a contradiction. ■

Lemma 1.9 *If $a, b \in X$ and $A, B \in P_{b,cl}(X)$, then $d(a, b) = H^\lambda(\{a\}, \{b\})$.*

Let $a, b \in X$ and $A, B \in P_{b,cl}(X)$. Then we have:

$$\begin{aligned} & H^\lambda(\{a\}, \{b\}) \\ &= \max\{H_\lambda(\{a\}, \{b\}), H_\lambda(\{b\}, \{a\})\} = \max\{(1 - \lambda)\rho(\{a\}, \{b\}) + \lambda\rho(\{b\}, \{a\}), \\ & (1 - \lambda)\rho(\{b\}, \{a\}) + \lambda\rho(\{a\}, \{b\})\} = \max\{(1 - \lambda)d(\{a\}, \{b\}) + \lambda d(\{b\}, \{a\}), \\ & (1 - \lambda)d(\{b\}, \{a\}) + \lambda d(\{a\}, \{b\})\} = \max\{d(a, b), d(a, b)\} = d(a, b). \end{aligned}$$

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