

Estimations of the norm of some linear operators involved in product quadrature formulas

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ABSTRACT. Two-sided estimations involving linear operators related to some product quadrature formulas of interpolatory type on Jacobi nodes are provided, in order to point out convergence theorems or the double superdense unbounded divergence concerning these formulas.

KEY WORDS: Product quadrature formulas, superdense sets, unbounded divergence, Jacobi nodes

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1 Introduction

Let $(C, \|\cdot\|)$ be the Banach space of all continuous functions $f : [-1, 1] \rightarrow \mathbb{R}$ endowed with the supremum norm $\|\cdot\|$, and let $\mathcal{P}_n$ be the space of all algebraic polynomials of degree at most n . Denote by $(L_1, \|\cdot\|_1)$ the Banach space of all measurable functions (equivalence classes of functions, with respect to the equality μ -a.e., where μ is the Lebesgue measure on

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$[-1, 1]$), normed by $\|g\|_1 = \int_{-1}^1 |g(x)|dx$. Analogously, $(L_\infty, \|\cdot\|_\infty)$ is the Banach space of all functions $g : [-1, 1] \rightarrow \mathbb{R}$ so that g is μ -a.e. bounded on $[-1, 1]$ and $\|g\|_\infty = \text{ess sup}|g|$. Given a nonnegative function $\rho \in L_\infty$ so that $\rho(x) > 0$ μ -a.e. on $[-1, 1]$, the notation $(L_1^{(1/\rho)}, \|\cdot\|_1^{(1/\rho)})$ stands for the Banach space of all measurable functions $g : [-1, 1] \rightarrow \mathbb{R}$ for which $g/\rho \in L_1$ and $\|g\|_1^{(1/\rho)} = \|g/\rho\|_1$, [5], [9].

Further, for each node matrix $\mathcal{M} = \{x_{kn} : n \geq 1, 1 \leq k \leq n\}$ with $-1 \leq x_{nn} < x_{n-1,n} < \dots < x_{2n} < x_{1n} \leq 1$, we denote by $L_n f \in \mathcal{P}_{n-1}$ and $\lambda_n : [-1, 1] \rightarrow [1, \infty)$, $n \geq 1$, the Lagrange polynomials and the Lebesgue functions associated to $\mathcal{M}$, respectively, namely

$$(L_n f)(x) = \sum_{k=1}^n f(x_{kn}) l_{kn}(x) \text{ and } \lambda_n(x) = \sum_{k=1}^n |l_{kn}(x)|, \quad |x| \leq 1,$$

where l_{kn} , $n \geq 1, 1 \leq k \leq n$, are the fundamental polynomials of Lagrange.

In this paper, we consider the product quadrature formulas (p.q.f.)

$$(1.1) \quad I(f; g) = I_n(f; g) + R_n(f; g); \quad n \geq 0,$$

where $I : X \times L_1^{(\rho)} \rightarrow \mathbb{R}$, $I_n : X \times L_1^{(\rho)} \rightarrow \mathbb{R}$,

$$(1.2) \quad I(f; g) = \int_{-1}^1 g(x) f(x) dx; \quad I_n(f; g) = \int_{-1}^1 g(x) (L_n f)(x) dx,$$

and $(X, \|\cdot\|_X)$, $X \subseteq C$, is a Banach space containing all the polynomials.

Throughout this paper, we suppose that the p.q.f. (1.1) are of interpolatory type, namely

$$(1.3) \quad I_n(P; g) = I(P; g), \quad \forall P \in \mathcal{P}_{n-1} \text{ and } g \in L_1^{(1/\rho)}; \quad n \geq 0.$$

We associate to the p.q.f. (1.1) the linear continuous operators

$$(1.4) \quad \mathcal{I}_n : X \rightarrow (L_1^{(1/\rho)})^*; \quad (\mathcal{I}_n f)(g) = I_n(f; g), \quad n \geq 0$$

and

$$(1.5) \quad \mathcal{I} : X \rightarrow (L_1^{(1/\rho)})^*; (\mathcal{I}f)(g) = I(f; g),$$

where Y^* is the space of all linear continuous functionals defined on some normed space Y and I_n, I are defined by (1.2).

The convergence of the p.q.f. (1.1) of interpolatory type was intensively studied for $X = C$ and various functions $g \in L_1$ and node matrices $\mathcal{M}$, see [2], [4], [5], [7], [9].

In accordance with [9], the convergence of the p.q.f. (1.1) for $X = C$ is equivalent to the boundedness of the set of norms $\{\|\mathcal{I}_n\| : n \geq 1\}$. The aim of this paper is precisely to provide estimations for the norms of the operators $\mathcal{I}_n$ and, consequently, to point out convergence and unbounded divergence results concerning p.q.f. (1.1) of interpolatory type for Jacobi node matrices. We will examine two situations regarding the Banach space $(X, \|\cdot\|_X)$: first, the classic space $(C, \|\cdot\|)$, and second, the space $(C^1, \|\cdot\|_{(1)})$, where C^1 is the set of all continuously differentiable functions $f : [-1, 1] \rightarrow \mathbb{R}$ and $\|f\|_{(1)} = \|f\| + \|f'\|$, $f \in C^1$.

In the following sections, we suppose that the node matrix $\mathcal{M}$ is the Jacobi node matrix $\mathcal{M}^{(\alpha, \beta)}$, with $\alpha > -1$, $\beta > -1$. Explicitly, denoting by $P_n^{(\alpha, \beta)}$, $n \geq 0$, the Jacobi polynomials, the nodes of the n -th row of $\mathcal{M}^{(\alpha, \beta)}$ are the roots $x_{kn}^{(\alpha, \beta)}$ of $P_n^{(\alpha, \beta)}$, with $x_{kn}^{(\alpha, \beta)} = \cos \theta_{kn}^{(\alpha, \beta)}$, $0 < \theta_{1,n}^{(\alpha, \beta)} < \theta_{2n}^{(\alpha, \beta)} < \dots < \theta_{nn}^{(\alpha, \beta)} < \pi$; for the sake of simplicity, we use the notations x_{kn} and θ_{kn} instead of $x_{kn}^{(\alpha, \beta)}$ and $\theta_{kn}^{(\alpha, \beta)}$, respectively; also, we set $x_{0n} = 1$ and $x_{n+1,n} = -1$.

Finally, we specify some notations to be used in this paper. The notation $[a]$ stands for the largest integer n with $n \leq a$, $a \in \mathbb{R}$. Also, we denote by m, M, M_k , $k \geq 1$, some generic positive constants which are independent of any positive integer n and we write $a_n \sim b_n$ if the sequences of real numbers (a_n) and (b_n) satisfy the conditions $b_n \neq 0$ and $0 < m \leq |a_n/b_n| \leq M$, $\forall n \geq 1$.

2 Product quadrature formulas involving continuous functions

In this section, we examine the p.q.f. (1.1) of interpolatory type for $X = C$. As we remarked in the introductory section, according to [9, (2.10)], the following characterization is valid:

$$(2.1) \quad \lim_{n \rightarrow \infty} I_n(f; g) = I(f; g), \quad \forall f \in C, \quad \forall g \in L_1^{(1/\rho)} \Leftrightarrow \limsup_{n \rightarrow \infty} \|\mathcal{I}_n\| < \infty.$$

Therefore, we begin by estimating the norm of the operators $\mathcal{I}_n$. Using standard arguments and the Theorem of Riesz concerning the representation of linear continuous functionals, we derive (see also [5] and [9]) the following formula:

$$(2.2) \quad \|\mathcal{I}_n\| = \sup\{\|\rho \cdot L_n f\|_\infty : f \in C, \|f\| \leq 1\}, \quad n \geq 0.$$

Now, we present a lower estimation involving the norm of $\mathcal{I}_n$.

Lemma 2.1 *Let n be a positive integer and suppose $\rho \in C$, $\rho(0) \neq 0$.*

1°. *If $\beta - \alpha \notin 2\mathbb{Z}$, then $\|\mathcal{I}_n\| \geq M \ln n$.*

2°. *If $\beta - \alpha \in 4\mathbb{Z}$, then $\|\mathcal{I}_{2n}\| \geq M \ln n$.*

3°. *If $\beta - \alpha \in 4\mathbb{Z} + 2$, then $\|\mathcal{I}_{2n+1}\| \geq M \ln n$.*

Proof. Let us define $f_n \in C$ by

$$f_n(x) = \begin{cases} \operatorname{sgn} l_{kn}(0), & \text{if } x = x_{kn}, \quad 0 \leq k \leq n+1 \\ \text{linear;} & \text{otherwise} \end{cases}$$

It follows from (2.2)

$$\|\mathcal{I}_n\| \geq \|\rho L_n f_n\|_\infty \geq \rho(0) |(L_n f_n)(0)| = \rho(0) \sum_{k=1}^n |l_{kn}(0)|,$$

so that

$$(2.3) \quad \|\mathcal{I}_n\| \geq \rho(0) |P_n^{(\alpha, \beta)}(0)| \sum_{k=1}^n \{|x_{kn}| \cdot |(P_n^{(\alpha, \beta)})'(x_{kn})|\}^{-1}.$$

Further, the relations

$$|P_{2n}^{(\alpha, \beta)}(0)| = \frac{1}{\sqrt{n\pi}} (\sqrt{2})^{\alpha+\beta} \left| (-1)^n \cos \frac{\beta - \alpha}{4} \pi + \frac{O(1)}{n} \right|$$

and

$$|P_{2n+1}^{(\alpha, \beta)}(0)| = \frac{1}{\sqrt{\pi(2n+1)}} (\sqrt{2})^{\alpha+\beta+1} \left| (-1)^{n+1} \sin \frac{\beta - \alpha}{4} \pi + \frac{O(1)}{n} \right|$$

follow from [10, Th. 8.21.13] and they lead to:

$$(2.4) \quad |P_{2n}^{(\alpha, \beta)}(0)| \sim 1/\sqrt{n}, \text{ if } \beta - \alpha \notin 4\mathbb{Z} + 2$$

$$(2.5) \quad |P_{2n+1}^{(\alpha, \beta)}(0)| \sim 1/\sqrt{n}, \text{ if } \beta - \alpha \notin 4\mathbb{Z}.$$

In order to estimate the roots $x_{kn} = \cos \theta_{kn}$, $1 \leq k \leq n$, of $P_n^{(\alpha, \beta)}$, we use the estimations $\theta_{kn} = \frac{1}{n}(k\pi + O(1))$, [10, formula (8.9.2)], which lead to the existence of an integer $s \geq 0$ so that

$$\theta_{kn} < \frac{\pi}{2}, \text{ if } 1 \leq k \leq \left\lfloor \frac{n}{2} \right\rfloor - s \text{ and } \theta_{kn} \geq \frac{\pi}{2}, \text{ if } \left\lfloor \frac{n}{2} \right\rfloor - s + 1 \leq k \leq n,$$

for sufficient large n . Applying [6, Lemma 2], which states that

$$|x - x_{kn}| \sim n^{-2} |m^2 - k^2|, \text{ if } x_{m+1, n} \leq x \leq x_{mn} \text{ and } k \notin \{m, m+1\},$$

we get for $m = \left\lfloor \frac{n}{2} \right\rfloor - s$ and $x = 0$

$$(2.6) \quad x_{kn} \sim \frac{1}{n} \left(\left\lfloor \frac{n}{2} \right\rfloor - s - k \right), \quad 1 \leq k \leq \left\lfloor \frac{n}{2} \right\rfloor - s - 1.$$

Also, the estimation

$$(2.7) \quad |(P_n^{(\alpha, \beta)})'(x_{kn})| \sim k^{-\alpha-3/2} n^{\alpha+2}, \text{ if } 0 < \theta_{kn} < \frac{\pi}{2}, \text{ [10]}$$

will be used in what follows.

Now, we examine the following situations.

First situation. Suppose $\beta - \alpha \notin 4\mathbb{Z} + 2$. Combining the relations (2.3), (2.4), (2.6) and (2.7), we obtain via Cauchy's inequality:

$$\begin{aligned} \|\mathcal{I}_{2n}\| &\geq M_1 \cdot \frac{1}{\sqrt{n}} \sum_{k=1}^{n-s-1} \frac{n}{n-s-k} \cdot \frac{k^{\alpha+3/2}}{n^{\alpha+2}} \\ &\geq M_1 \cdot n^{-\alpha-1/2} \left(\frac{1}{n} \sum_{k=1}^{n-s-1} k^{\alpha+3/2} \cdot \frac{1}{n-s-k} \right) \\ &\geq M_2 n^{-\alpha-1/2} n^{-2} (n-s-1)^{\alpha+5/2} \ln(n-s) \\ &\geq M_2 \left(\frac{n-s-1}{n} \right)^{\alpha+5/2} \ln(n-s), \end{aligned}$$

i.e. $\|\mathcal{I}_{2n}\| \geq M_3 \ln n$, for sufficient large n .

Second situation. Suppose $\beta - \alpha \notin 4\mathbb{Z}$. Similarly to the first situation, we get from (2.3), (2.5), (2.6), (2.7) and Cauchy's inequality, $\|\mathcal{I}_{2n+1}\| \geq M_4 \ln n$, for sufficient large n .

Finally, the assertions of this theorem follow, remarking that the hypothesis $\beta - \alpha \notin 2\mathbb{Z}$ of 1° means that both the first and the second situations hold; similarly 2° implies the first situation and 3° implies the second one.

Remark 2.1 *The estimations of Lemma 2.1 show that $\limsup_{n \rightarrow \infty} \|\mathcal{I}_n\| = \infty$, for each $\alpha > -1$ and $\beta > -1$, if $\rho \in C$ and $\rho(0) \neq 0$. However, the conditions $\rho \in C$ and $\rho(0) \neq 0$, which hold in the classic examples, are not essential for the unboundedness of the set $\{\|\mathcal{I}_n\| : n \geq 1\}$. Indeed, since $\rho(x) > 0$ μ -a.e. on $[-1, 1]$, there exists $x_0 \in (-1, 1)$ so that $\rho(x_0) > 0$ and we derive from (2.2), $\|\mathcal{I}_n\| \geq \rho(x_0) \lambda_n(x_0)$. Next, using [10, (14.4.7)], a strictly increasing sequence (j_n) of positive integers exists so that $\lambda_{j_n}(x_0) \sim \ln j_n$, which implies $\limsup_{n \rightarrow \infty} \|\mathcal{I}_n\| = \infty$. We remark that the estimations of Lemma 2.1 are more accurate.*

Now, the assertion (2.1) and Remark 2.1 show the impossibility of the convergence of p.q.f. (1.1) for all $f \in C$ and $g \in L_1^{(1/\rho)}$, if $\alpha > -1$ and $\beta > -1$ are given arbitrarily.

In fact, these quadrature formulas are unboundedly divergent on large subsets of C and $L_1^{(1/\rho)}$, in topological sense, and this divergence holds for arbitrary polynomial projection operators $\mathcal{L}_n$ instead of Lagrange projections L_n (namely $\mathcal{L}_n : C \rightarrow \mathcal{P}_{n-1}$ so that $\mathcal{L}_n f = f$ if and only if $f \in \mathcal{P}_{n-1}$) and arbitrary weight functions $\rho \in L_\infty$ such that ρ is nonnegative and $\rho(x) > 0$ μ -a.e. on $[-1, 1]$, as follows.

Theorem 2.1 [5] *There exists a superdense set X_0 in the Banach space $L_1^{(1/\rho)}$ such that for every g in X_0 , the subset of C consisting of all functions f for which the product quadrature formulas (1.1) of interpolatory type are unboundedly divergent, namely*

$$Y_0(g) = \left\{ f \in C : \limsup_{n \rightarrow \infty} \left| \int_{-1}^1 g(x)(\mathcal{L}_n f)(x) dx \right| = \infty \right\}$$

is superdense in the Banach space C .

We recall that a subset S of a topological space T is said to be *superdense* in T , if it is residual (i.e. its complement is of first Baire category), uncountable and dense in T .

Further, in order to obtain convergence results, we will derive upper estimations for $\|\mathcal{I}_n\|$. Firstly, we remark that

$$(2.8) \quad \begin{aligned} |\rho(x)(L_n f)(x)| &= \rho(x) \left| \sum_{k=1}^n f(x_{kn}) l_{kn}(x) \right| \\ &\leq \|\rho\|_\infty \lambda_n(x) \|f\|; \quad |x| \leq 1, \quad f \in C. \end{aligned}$$

Let $q = \max\{\alpha, \beta\}$. The relations

$$\lambda_n(x) = \begin{cases} O(\ln(2 + n\theta)), & 0 \leq x = \cos \theta \leq 1, \quad \alpha \leq -\frac{1}{2} \\ O(\ln n + |P_n^{(\alpha, \beta)}(x)|\sqrt{n}), & 0 \leq x \leq 1, \quad \alpha > -\frac{1}{2} \end{cases} ; [1],$$

$$\max\{|P_n^{(\alpha, \beta)}(x)| : |x| \leq 1\} \sim n^q, \text{ if } q > -\frac{1}{2}, [10],$$

and

$$P_n^{(\alpha, \beta)}(x) = (-1)^n P_n^{(\beta, \alpha)}(-x), \quad |x| \leq 1, \quad [10],$$

combined with (2.8), give:

$$(2.9) \quad \begin{cases} \|\mathcal{I}_n\| \leq M_5 \ln n, & \text{if } q \leq -\frac{1}{2} \\ \|\mathcal{I}_n\| \leq M_6 n^{q+1/2}, & \text{if } q > -\frac{1}{2} \end{cases}, \quad \text{for } n \text{ sufficient large.}$$

Denote by $\omega(f; \delta)$ the modulus of continuity of a function $f \in C$ and let $Lip\gamma$, $0 < \gamma \leq 1$, be the set of all functions $f \in C$ satisfying the inequality $\omega(f; \delta) \leq M\delta^\gamma$, $0 \leq \delta \leq 2$. Now, we can state the following convergence result.

Theorem 2.2 *The product quadrature formulas (1.1) of interpolatory type are convergent for each $g \in L_1^{(1/\rho)}$ in the following situations:*

1°. $q \leq -\frac{1}{2}$ and $f \in C$ satisfies a Dini-Lipschitz condition

$$\lim_{\delta \searrow 0} \omega(f; \delta) \ln \delta = 0.$$

2°. $q > -\frac{1}{2}$ and $f \in C$ satisfies the condition $\omega(f; \delta) \in \sigma(\delta^{q+1/2})$.

3°. $q \in \left(-\frac{1}{2}, \frac{1}{2}\right)$, $f \in Lip\gamma$ and $q + \frac{1}{2} < \gamma \leq 1$.

Proof. Let us estimate the errors $R_n(f; g)$ of p.q.f. (1.1). We get, for arbitrary $f \in C$ and $P \in \mathcal{P}_{n-1}$:

$$\begin{aligned} |R_n(f; g)| &= |(I_n - I)(f - P; f)| = |(\mathcal{I}_n - \mathcal{I})(f - P)(g)| \\ &\leq \|\mathcal{I}_n - \mathcal{I}\| \cdot \|f - P\| \cdot \|g\|_1^{(1/\rho)}, \end{aligned}$$

i.e.

$$(2.10) \quad |R_n(f; g)| \leq \|g\|_1^{(1/\rho)} (\|\mathcal{I}_n\| + \|\rho\|_\infty) \|f - P\|; \quad P \in \mathcal{P}_{n-1}, \quad f \in C.$$

Denoting by $E_{n-1}(f) = \text{dist}(f; \mathcal{P}_{n-1}) = \inf\{\|f - P\| : P \in \mathcal{P}_{n-1}\}$, $f \in C$,

we get, using (2.9), (2.10) and the inequality $E_{n-1}(f) \leq M\omega\left(f; \frac{1}{n}\right)$, [8]:

$$|R_n(f, g)| \leq \begin{cases} M_7(\|\rho\|_\infty + \ln n)\omega\left(f; \frac{1}{n}\right)\|g\|_1^{(1/\rho)}, & \text{if } q \leq -\frac{1}{2} \\ M_8(\|\rho\|_\infty + n^{q+1/2})\omega\left(f; \frac{1}{n}\right)\|g\|_1^{(1/\rho)}, & \text{if } q > \frac{1}{2}, \end{cases}$$

for sufficient large n , which completes the proof.

3 Product quadrature formulas involving continuously differentiable functions

In this section, we consider the p.q.f. (1.1) of interpolatory type for $X = C^1$. Using (2.9), (2.10) and the inequality $E_{n-1}(f) \leq M \cdot \frac{1}{n}\omega\left(f'; \frac{1}{n}\right)$, [8], we obtain:

$$|R_n(f; g)| \leq \begin{cases} M_9\left(1 + \frac{\ln n}{n}\right)\omega\left(f'; \frac{1}{n}\right)\|g\|_1^{(1/\rho)}, & \text{if } q \leq -\frac{1}{2} \\ M_{10}(1 + n^{q-1/2})\omega\left(f'; \frac{1}{n}\right)\|g\|_1^{(1/\rho)}, & \text{if } q > -\frac{1}{2}, \end{cases}$$

for $f \in C^1$, $g \in L_1^{(1/\rho)}$ and sufficient large n . Therefore, we can state the following convergence result.

Theorem 3.1 *The product quadrature formulas (1.1) of interpolatory type are convergent for each $g \in L_1^{(1/\rho)}$ in the following situations:*

- 1°. $q \leq -\frac{1}{2}$ and $f \in C^1$.
- 2°. $q > -\frac{1}{2}$ and $f \in C^1$ satisfies the relation $\omega(f'; \delta) = \sigma(\delta^{q-1/2})$.
- 3°. $q \in \left(-\frac{1}{2}, \frac{3}{2}\right)$, $f' \in Lip\gamma$ and $q - \frac{1}{2} < \gamma \leq 1$.

Now, it is obvious that an unbounded divergence result, similar to that stated by Theorem 2.1, fails for $X = C^1$ and arbitrary $\alpha > -1$, $\beta > -1$. However, if $q > \frac{1}{2}$, then the double superdense unbounded divergence of p.q.f. (1.1) can occur, as we will show in the next statement, for a particular choice of the weight function ρ .

Theorem 3.2 *Suppose that $\rho(x) = (1-x)^a(1+x)^b$, $x \in [-1, 1]$, $a \geq 0$, $b \geq 0$, and $\alpha > \frac{1}{2} + 2a$ or $\beta > \frac{1}{2} + 2b$. Then, there exists a superdense set X_0 in the Banach space $L_1^{(1/\rho)}$ such that for every g in X_0 the subset of C^1 consisting of all functions for which the p.q.f. (1.1) of interpolatory type are unboundedly divergent, i.e.*

$$Y_0(g) = \left\{ f \in C^1 : \limsup_{n \rightarrow \infty} \left| \int_{-1}^1 g(x)(L_n f)(x) dx \right| = \infty \right\},$$

is superdense in the Banach space C^1 .

Proof. For the sake of simplicity, in this proof we set x_k, θ_k, l_k instead of x_{kn}, θ_{kn} and l_{kn} , respectively. The norm of corresponding operator $\mathcal{I}_n : C^1 \rightarrow \left(L_1^{(1/\rho)}\right)^*$ is given by:

$$(3.1) \quad \|\mathcal{I}_n\| = \sup\{\|\rho L_n f\|_\infty : f \in C^1, \|f\|_{(1)} \leq 1\}, \quad n \geq 0,$$

which is a similar formula to (2.2). Let us define $h_n \in C^1$ as follows:

$$\begin{aligned} h_n(x) &= -1, \text{ if } x_1 < x \leq 1; \\ h_n(x) &= (-1)^n \cdot n, \text{ if } -1 \leq x \leq x_n; \\ h_n(x) &= \frac{1}{4}(-1)^{k+1} \left\{ 2 + (6k+3) \left(\frac{2}{x_k - x_{k+1}} \right)^3 \left(x - \frac{x_k + x_{k+1}}{2} \right) \right. \\ &\quad \left. \times \left(\frac{1}{3} \left(x - \frac{x_k + x_{k+1}}{2} \right)^2 - \left(\frac{x_k - x_{k+1}}{2} \right)^2 \right) \right\}, \end{aligned}$$

if $x_{k+1} < x \leq x_k$, $1 \leq k \leq n-1$, $n \geq 2$.

In order to estimate $\|h_n\|_{(1)}$, we notice that $\theta_{k+1} - \theta_k \sim \frac{1}{n}$, $\sin \theta_k \sim \frac{k}{n}$, and $\theta \sim \theta_k$ if $\theta_k \leq \theta \leq \theta_{k+1}$, [7], which implies

$$x_k - x_{k+1} = 2 \sin \frac{\theta_{k+1} - \theta_k}{2} \sin \frac{\theta_k + \theta_{k+1}}{2} \sim \frac{k}{n^2}.$$

Now, standard calculations give:

$$\max\{|h_n(x)| : x_k \leq x \leq x_{k+1}\} = k + 1,$$

$$\max\{|h'_n(x)| : x_k \leq x \leq x_{k+1}\} \leq M_{11}n^2,$$

hence $\|h_n\|_{(1)} \leq M_{12}n^2$. Denoting

$$f_n = \frac{1}{M_{12}n^2}h_n,$$

we obtain from (3.1):

$$(3.2) \quad \|\mathcal{I}_n\| \geq \|\rho L_n f_n\|.$$

Next, using [10, formula (8.21.18)] for $\theta = \frac{c}{n}$, $c > 0$, we get:

$$(3.3) \quad P_n^{(\alpha, \beta)}\left(\cos \frac{c}{n}\right) = \frac{1}{\sqrt{n\pi}} \left(\sin \frac{c}{2n}\right)^{-\alpha-1/2} \left(\cos \frac{c}{2n}\right)^{-\beta-1/2} \cdot \zeta_n,$$

where

$$\zeta_n = \cos\left(c - \frac{2\alpha+1}{4}\pi + \frac{\alpha+\beta+1}{2n}c\right) + O(1) \left(n \sin \frac{c}{n}\right)^{-1}.$$

It exists $c > 0$ with $|\zeta_n| \geq M_{13} > 0$ for sufficient large n , so that combining (3.3) and the relations $\sin \frac{c}{2n} \sim n^{-1}$, $\cos \frac{c}{2n} \sim 1$, we derive:

$$(3.4) \quad \left|P_n^{(\alpha, \beta)}\left(\cos \frac{c}{n}\right)\right| \geq M_{14}n^{-\alpha}, \text{ for sufficient large } n.$$

Further,

$$\rho\left(\cos \frac{c}{n}\right) \geq \left(1 - \cos \frac{c}{n}\right)^a = \left(2 \sin^2 \frac{c}{2n}\right)^a,$$

therefore

$$(3.5) \quad \rho\left(\cos \frac{c}{n}\right) \sim n^{-2a}.$$

Now, the relations (2.6), (2.7), (3.2), (3.3), (3.4) and (3.5) give:

$$\begin{aligned} \|\mathcal{I}_{2n}\| &\geq \rho\left(\cos \frac{c}{2n}\right) \left| (L_{2n} f_{2n})\left(\cos \frac{c}{2n}\right) \right| \\ &= \rho\left(\cos \frac{c}{2n}\right) \left| \sum_{k=1}^{2n} f_{2n}(x_k) l_k\left(\cos \frac{c}{2n}\right) \right| \\ &\geq M_{15} n^{-2a-2} \sum_{k=1}^{n-s-1} k n^\alpha \left| \cos \frac{c}{2n} - \cos \theta_k \right|^{-1} k^{\alpha+3/2} n^{-\alpha-2} \\ &\geq M_{16} n^{-2a-2} (n-s-1)^{\alpha+3/2}, \end{aligned}$$

i.e.

$$(3.6) \quad \|\mathcal{I}_{2n}\| \geq M_{17} n^{\alpha-1/2-2a}, \text{ for sufficient large } n,$$

taking into account that

$$\left| \cos \frac{c}{2n} - \cos \theta_k \right| \leq 1 - \cos \theta_k = 2 \sin^2 \left(\frac{\theta_k}{2} \right) \sim k^2 n^{-2}.$$

Similarly to (3.6), using classic arguments, we obtain:

$$(3.7) \quad \|\mathcal{I}_{2n+1}\| \geq n^{\beta-1/2-2b}, \text{ for sufficient large } n.$$

The inequalities (3.6) and (3.7) show that the set of norms $\{\|\mathcal{I}_n\| : n \geq 1\}$ is unbounded, if $\alpha > \frac{1}{2} + 2a$ or $\beta > \frac{1}{2} + 2b$. Applying the principle of condensation of singularities [3, Theorem 5.4] and the principle of double condensation of singularities [3, Theorem 5.2], in a similar manner to [5, Theorem 2.1], we complete the proof.

References

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