

Fixed points theorems of generalized $\alpha - \psi$ -contractive type operators

ANCA MARIA OPREA
(CLUJ-NAPOCA)

ABSTRACT. In this paper we shall extend the result of B. Samet, C. Vetro and P. Vetro [12], J.Hasanzade, S. Rezapour and N. Shazad [1] and some results from papers [5], [11], [12], [13] about $\alpha - \psi$ -contractive and α -Geraghty contractions for single-valued and multi-valued mappings (see [14]). The aim of this paper is to modify further the notions of $\alpha - \psi$ -contractive, α -Geraghty contraction type and α -admissible mappings and establish fixed point theorems for such mappings in complete metric spaces.

KEY WORDS: fixed point, multivalued operator, α -admissible operators, $\alpha - \psi$ -contractive mappings, α -Geraghty contractions.

MSC 2010: 47H10, 54H25

1 Introduction

In 1973, Geraghty [3] give a generalization of the contraction principle, by introducing an interesting class of auxiliary functions $\alpha: \mathbb{R}_+ \rightarrow [0, 1)$ with the property:

[◇]Anca Maria Oprea, Babeş-Bolyai University,
Faculty of Mathematics and Computer Sciences,
email: anca.oprea@math.ubbcluj.ro

If $\alpha(t_n) \rightarrow 1$, then $t_n \rightarrow 0$ (as $n \rightarrow +\infty$).

Recently B. Samet, C. Vetro, P. Vetro [12] introduced the class of $\alpha - \psi$ -contractive type mappings and proved some new fixed point theorems. More recently E. Karapinar and B. Samet [6] generalize the above results for the class of $\alpha - \psi$ -contractive mappings of Ćirić type.

In 2014, N. Hussain, M.A. Kutbi and P. Salimi [5] gave some fixed point theorems for $\alpha - \psi$ -contractive mappings satisfying a rational contractive condition. An application to integral equations is also proved.

The purpose of this paper is to prove some fixed point theorems for multivalued $\alpha - \psi$ -contractive mappings satisfying some rational contractive assumptions.

We need to recall some notions (see [2], [10]) and auxiliary results.

Definition 1.1 Let Ψ be the family of functions $\psi: [0, +\infty) \rightarrow [0, +\infty)$ satisfying the following conditions:

- i) ψ is nondecreasing;
- ii) $\sum_{n=1}^{+\infty} \psi^n(t) < \infty$ for all $t > 0$, where $\psi^n(t)$ is the n th iterate of ψ .

These functions are known in the literature as a strong comparison function. It is easily to see that if ψ is a strong comparison function, then $\psi(t) < t$ for any $t > 0$.

Definition 1.2 ([12]) Let (X, d) be a metric space, $T: X \rightarrow X$ and $\alpha: X \times X \rightarrow [0, +\infty)$. We say that T is α -admissible if $x, y \in X$, $\alpha(x, y) \geq 1 \Rightarrow \alpha(Tx, Ty) \geq 1$.

Definition 1.3 ([12]) Let (X, d) be a complete metric space and $T: X \rightarrow X$ be a given mapping. We say that T is an $\alpha - \psi$ -contractive mapping if there exists two functions $\alpha: X \times X \rightarrow [0, +\infty)$ and $\psi \in \Psi$ such that

$$\alpha(x, y)d(Tx, Ty) \leq \psi(d(x, y)) \quad \text{for all } x, y \in X.$$

The main result of the paper (see [12]) is the following theorem.

Theorem 1.1 ([12]) *Let (X, d) be a complete metric space and $T: X \rightarrow X$ be an $\alpha - \psi$ -contractive mapping. Suppose that:*

- i) T is admissible;*
- ii) there exists $x_0 \in X$ such that $\alpha(x_0, Tx_0) \geq 1$;*
- iii) T is continuous.*

Then, T has a fixed point that is, there exists $x^* \in X$ such that $Tx^* = x^*$.

In 2012 J. Hasanzade, S. Rezapour and N. Shahzad results [4] introduced for the first time the notion of α_* - ψ -contractive multivalued mappings.

Definition 1.4 ([4]) *If (X, d) be a metric space, and $A, B \in P(X)$, then $\alpha_*(A, B) := \inf\{\alpha(a, b) : a \in A, b \in B\}$. Let $T: X \rightarrow P(X)$, we say that T is α_* -admissible whenever $\alpha(x, y) \geq 1$ implies $\alpha_*(Tx, Ty) \geq 1$.*

Definition 1.5 ([4]) *Let (X, d) be a complete metric space, $T: X \rightarrow P_{cl}(X)$ be a closed multi-valued mapping, $\psi \in \Psi$ and $\alpha: X \times X \rightarrow [0, \infty)$ be a function. We say that T is an $\alpha_* - \psi$ -contractive multi-valued mapping whenever*

$$\alpha_*(Tx, Ty)H(Tx, Ty) \leq \psi(d(x, y))$$

for $x, y \in X$.

In 2012, E. Karapinar, B. Samet [6] introduced the concept of triangular α -admissible mappings, and after that many papers related to this subject appeared. One of this papers is Ovidiu Popescu [9], which appeared in 2014, where the main result of it is based on the following concept.

Definition 1.6 *An α -admissible map T is said to be triangular α -admissible if*

- i) $x, y \in X, \alpha(x, y) \geq 1 \Rightarrow \alpha(Tx, Ty) \geq 1$;*
- ii) $x, y, z \in X, \alpha(x, z) \geq 1$ and $\alpha(z, y) \geq 1 \Rightarrow \alpha(x, y) \geq 1$.*

Remark 1.1 Let $T: X \rightarrow X$ be a triangular α -admissible map. Assume that there exists $x_0 \in X$ such that $\alpha(x_0, Tx_0) \geq 1$. If we define a sequence $\{x_n\}$ by $x_{n+1} \in Tx_n$, then we have $\alpha(x_n, Tx_m) \geq 1$ for all $m, n \in \mathbb{N}$ with $n < m$.

Definition 1.7 We denote $\mathfrak{F}$ the family of all functions $\beta: [0, \infty) \rightarrow [0, 1)$ which satisfies the condition: $\lim_{n \rightarrow \infty} \beta(t_n) = 1$ implies $\lim_{n \rightarrow \infty} t_n = 0$.

Definition 1.8 Let (X, d) be a complete metric space, and let $\alpha: X \times X \rightarrow \mathbb{R}$ be a function. A map $T: X \rightarrow P(X)$ is called a generalized α -Geraghty contraction type map if there exists $\beta \in \mathfrak{F}$ such that for all $x, y \in X$

$$\alpha(x, y)d(Tx, Ty) \leq \beta(M(x, y)) \cdot M(x, y),$$

where

$$M(x, y) = \max\left\{d(x, y), \frac{d(y, Ty)[1 + d(x, Tx)]}{1 + d(x, y)}\right\}.$$

Lemma 1.1 Let $q > 1$ $A, B \in P(X)$. Then for each $a \in A$ there exists $b \in B$ such that $d(a, b) \leq qH(A, B)$.

If Y is a nonempty subset of X and $T: Y \rightarrow P(X)$ is a multivalued operator, then by definition, an element $x \in Y$ is said to be:

- (i) a fixed point of T if and only if $x \in T(x)$;
- (ii) a strict fixed point of T if and only if $x = T(x)$.

We will denote by F_T the fixed point set for T and by SF_T the strict fixed point set of T .

2 Generalized $\alpha - \psi$ -contractive mappings

In the first part of the paper we will extend J. Hasanzade, S. Rezapour and N. Shahzad results [4], for the case of $\alpha - \psi$ -contractive multi-valued mapping, by using a weaker contractive condition (see [8]).

Theorem 2.1 *Let (X, d) be a complete metric space and $T: X \rightarrow P_{cl}(X)$ be multi-valued mapping. Suppose that there exists $\alpha: X \times X \rightarrow \mathbb{R}$ and $\psi: \mathbb{R}_+ \rightarrow \mathbb{R}_+$ such that:*

i) the inequality holds

$$\alpha_*(x, y)H(Tx, Ty) \leq ad(x, y) + b \frac{D(y, Ty)[1 + D(x, Tx)]}{1 + d(x, y)}$$

where $a, b > 0$ with $a + b < 1$;

ii) T is α_* -admissible;

iii) There exists $x_0 \in X$ and $x_1 \in Tx_0$ such that $\alpha(x_0, x_1) \geq 1$;

iv) if $\{x_n\}$ is a sequence in X such that $\alpha(x_n, x_{n+1}) \geq 1$ for all $n \in \mathbb{N}$ and $x_n \rightarrow x$, then $\alpha(x_n, x) \geq 1$ for all $n \in \mathbb{N}$.

v) T is continuous.

Then T has a fixed point.

Proof. If $x_0 = x_1$ then we have nothing to prove. So let $x_0 \neq x_1$. If $x_1 \in Tx_0$, and consider $1 < q < \frac{1}{a+b}$ then by Lemma 1.1, there exists $x_2 \in Tx_1$ with $0 < d(x_1, x_2) \leq qH(Tx_0, Tx_1)$, and if we use that T is α_* -admissible we get

$$\begin{aligned} 0 < d(x_1, x_2) &\leq q\alpha_*(Tx_0, Tx_1)H(Tx_0, Tx_1) \\ &\leq q \left[ad(x_0, x_1) + b \frac{D(x_1, Tx_1)[1 + D(x_0, Tx_0)]}{1 + d(x_0, x_1)} \right] \end{aligned}$$

Since $x_1 \in Tx_0$, then we have

$$\begin{aligned} 0 < d(x_1, x_2) &\leq q \left[ad(x_0, x_1) + b \frac{d(x_1, x_2)[1 + d(x_0, x_1)]}{1 + d(x_0, x_1)} \right] \\ &\leq q[ad(x_0, x_1) + bd(x_1, x_2)] \end{aligned}$$

Hence, we obtain $d(x_1, x_2)(1 - q \cdot b) \leq q \cdot a \cdot d(x_0, x_1)$ and so

$$d(x_1, x_2) \leq \frac{qa}{1 - qb} d(x_0, x_1).$$

We notice that $\frac{qa}{1 - qb} := \beta < 1$.

If $x_2 \in Tx_2$, then x_2 is a fixed point of T . Assume that $x_2 \neq Tx_2$ and $q > 1$, then

$$\begin{aligned} 0 < d(x_2, x_3) &\leq q\alpha_*(Tx_1, Tx_2)H(Tx_1, Tx_2) \\ &\leq q \left[ad(x_1, x_2) + b \frac{D(x_2, Tx_2)[1 + D(x_1, Tx_1)]}{1 + d(x_1, x_2)} \right] \end{aligned}$$

Since $x_2 \in Tx_1$, then we have

$$\begin{aligned} 0 < d(x_2, x_3) &\leq q \left[ad(x_1, x_2) + b \frac{d(x_2, x_3)[1 + d(x_1, x_2)]}{1 + d(x_1, x_2)} \right] \\ &\leq q[ad(x_1, x_2) + bd(x_2, x_3)] \end{aligned}$$

So we have

$$d(x_2, x_3) \leq \frac{qa}{1 - qb} d(x_1, x_2) \leq \left(\frac{qa}{1 - qb} \right)^2 d(x_0, x_1) = \beta^2 d(x_0, x_1).$$

By the same procedure we obtain a sequence $\{x_n\} \in X$ such that $x_n \in Tx_{n-1}$, $x_n \neq x_{n-1}$, $\alpha(x_n, x_{n+1}) \geq 1$ and $d(x_n, x_{n+1}) \leq \beta^n d(x_0, x_1)$ for all $n \in \mathbb{N}$.

Then, for each $n < m$ we have

$$\begin{aligned} d(x_n, x_m) &\leq d(x_{n-1}, x_n) + \dots + d(x_{m-1}, x_m) \leq (\beta^n + \dots + \beta^{m-1})d(x_0, x_1) \\ &\leq \frac{\beta^n}{1 - \beta} d(x_0, x_1) \rightarrow 0 \quad \text{as } n, m \rightarrow \infty. \end{aligned}$$

and so $\{x_n\}$ is a Cauchy sequence in X . Hence there exists $x^* \in X$ such that $x_n \rightarrow x^*$. Since $\alpha(x_n, x^*) \geq 1$ for all n and T is α_* -admissible,

$\alpha_*(Tx_n, Tx^*) \geq 1$ for all n , thus

$$\begin{aligned}
D(x^*, Tx^*) &= \inf_{y \in Tx^*} d(x^*, y) \\
&\leq d(x^*, x_{n+1}) + \inf_{y \in Tx^*} d(x_{n+1}, y) \\
&\leq d(x^*, x_{n+1}) + \alpha_*(Tx_n, Tx^*)H(Tx_n, Tx^*) \\
&\leq d(x^*, x_{n+1}) + ad(x_n, x^*) + b \cdot \frac{D(x^*, Tx^*)[1 + D(x_n, Tx_n)]}{1 + d(x_n, x^*)} \\
&\leq d(x^*, x_{n+1}) + ad(x_n, x^*) + b \cdot \frac{D(x^*, Tx^*)[1 + d(x_n, x_{n+1})]}{1 + d(x_n, x^*)} \\
&\rightarrow bD(x^*, Tx^*).
\end{aligned}$$

Hence $D(x^*, Tx^*) = 0$ and so $x^* \in F_T$. ■

Remark 2.1 Under similar assumptions as in Theorem 2.1, we can obtain a similar result for the case of single-valued mappings.

Corollary 2.1 Let (X, d) be a complete metric space and $T: X \rightarrow X$ be an $\alpha - \psi$ -contractive mapping. Suppose that there exists $\alpha: X \times X \rightarrow \mathbb{R}_+$, such that

i) the inequality $\alpha(x, y)d(Tx, Ty) \leq ad(x, y) + b \frac{d(y, Ty)[1 + d(x, Tx)]}{1 + d(x, y)}$ holds for all $x, y \in X$;

ii) T is α -admissible;

iii) there exists $x_0 \in X$ such that $\alpha(x_0, Tx_0) \geq 1$;

iv) T is continuous

Then, T has a fixed point $x^* \in X$.

Our next result is based on the concept of $\alpha - \psi$ -contractive type mapping (see B. Samet, C. Vetro, P. Vetro [12]), but we use here a contractive condition of rational type.

Theorem 2.2 *Let (X, d) be a complete metric space and $T: X \rightarrow X$ be $\alpha - \psi$ -contractive single-valued mapping. Suppose that there exists $\alpha: X \times X \rightarrow \mathbb{R}$ and $\psi: \mathbb{R}_+ \rightarrow \mathbb{R}_+$ be an increasing map such that:*

i) the inequality holds

$$\alpha(x, y)d(Tx, Ty) \leq \psi \left(\max \left\{ d(x, y), \frac{d(y, Ty)[1 + d(x, Tx)]}{1 + d(x, y)} \right\} \right);$$

ii) T is α -admissible;

iii) There exists $x_0 \in X$ and $x_1 \in Tx_0$ such that $\alpha(x_0, x_1) \geq 1$;

iv) T is continuous.

Then, T has a fixed point $x^* \in X$.

Proof. Let $x_0 \in X$ and $x_1 = Tx_0$ such that $\alpha(x_0, Tx_0) \leq 1$. If $x_0 = x_1$, than the proof is finished.

Suppose that $x_0 \neq x_1$. Then we have

$$\begin{aligned} d(x_1, Tx_1) &\leq \alpha(x_0, x_1)d(Tx_0, Tx_1) \\ &\leq \psi \left(\max \left\{ d(x_0, x_1), \frac{d(x_1, Tx_1)[1 + d(x_0, Tx_0)]}{1 + d(x_0, x_1)} \right\} \right) \\ &= \psi \left(\max \left\{ d(x_0, x_1), \frac{d(x_1, Tx_1)[1 + d(x_0, x_1)]}{1 + d(x_0, x_1)} \right\} \right) \\ &= \psi(\max\{d(x_0, x_1), d(x_1, Tx_1)\}) \end{aligned}$$

If $\max\{d(x_0, x_1), d(x_1, Tx_1)\} = d(x_1, Tx_1)$,

then $d(x_1, Tx_1) \leq \psi(d(x_1, Tx_1))$. So we get that $d(x_1, Tx_1) = 0$, which is a contradiction. Hence, we get that $d(x_1, Tx_1) \leq \psi(d(x_0, x_1))$. If $x_1 = Tx_1$, then $x_1 \in F_T$. Let $x_1 \neq Tx_1$ and $q > 1$. Then

$$0 < d(x_1, Tx_1) < q\psi(d(x_0, x_1)).$$

Choose $t_0 = d(x_0, x_1)$. Then $t_0 > 0$ and $d(x_1, Tx_1) < q\psi(t_0)$. Hence there exists $x_2 = Tx_1$ such that $d(x_1, x_2) < q\psi(t_0)$ and so $\psi(d(x_1, x_2)) < \psi(q\psi(t_0))$. It is clear that $x_2 \neq x_1$. Put $q_1 = \frac{\psi(q\psi(t_0))}{\psi(d(x_1, x_2))}$. Then $q_1 > 1$ and

we have

$$\begin{aligned}
d(x_2, Tx_2) &\leq \alpha(x_1, x_2)d(Tx_1, Tx_2) \\
&\leq \psi \left(\max \left\{ d(x_1, x_2), \frac{d(x_2, Tx_2)[1 + d(x_1, Tx_1)]}{1 + d(x_1, x_2)} \right\} \right) \\
&= \psi \left(\max \left\{ d(x_1, x_2), \frac{d(x_2, Tx_2)[1 + d(x_1, x_2)]}{1 + d(x_1, x_2)} \right\} \right) \\
&= \psi(\max\{d(x_1, x_2), d(x_2, Tx_2)\})
\end{aligned}$$

By the same procedure we choose $\max\{d(x_1, x_2), d(x_2, Tx_2)\} = d(x_1, x_2)$ and we get $d(x_2, Tx_2) \leq \psi(d(x_1, x_2))$.

If $x_2 = Tx_2$, then we have $x_2 \in F_T$. Let $x_2 \neq Tx_2$. So we get $0 < d(x_2, Tx_2) < q_1\psi(d(x_1, x_2))$. Hence there exists $x_3 = Tx_2$ such that $d(x_2, x_3) < q_1\psi(d(x_1, x_2)) = \psi(q\psi(t_0))$. So it is clear that $x_2 \neq x_3$ and $\psi(d(x_2, x_3)) < \psi^2(q\psi(t_0))$. Choose $q_2 = \frac{\psi^2(q\psi(t_0))}{\psi(d(x_2, x_3))}$. Then $q_2 > 1$ and we have

$$\begin{aligned}
d(x_3, Tx_3) &\leq \alpha(x_2, x_3)d(Tx_2, Tx_3) \leq \\
&\leq \psi \left(\max \left\{ d(x_2, x_3), \frac{d(x_3, Tx_3)[1 + d(x_2, Tx_2)]}{1 + d(x_2, x_3)} \right\} \right) \\
&= \psi \left(\max \left\{ d(x_2, x_3), \frac{d(x_3, Tx_3)[1 + d(x_2, x_3)]}{1 + d(x_2, x_3)} \right\} \right) \\
&= \psi(\max\{d(x_2, x_3), d(x_3, Tx_3)\})
\end{aligned}$$

If we continue by the same procedure we obtain a sequence $\{x_n\} \in X$ such that $x_n = Tx_{n-1}$, $x_n \neq x_{n-1}$ and $d(x_n, x_{n+1}) < \psi^{n-1}(q\psi(t_0))$. Let $m > n$. Then

$$\psi(d(x_n, x_m)) \leq \sum_{i=n}^{m-1} \psi^{i-1}(q\psi(t_0))$$

and this proves that $\{x_n\}$ is a Cauchy sequence. Hence there exists $x^* \in X$ such as $x_n \rightarrow x^*$. We know that T is continuous, then

$$d(x^*, Tx^*) = \lim_{n \rightarrow \infty} d(x_{n+1}, Tx^*) = \lim_{n \rightarrow \infty} d(Tx_n, Tx^*) = 0.$$

So we get that $x^* \in F_T$. ■

Using this assumptions we can build an example which verifies the Theorem 2.2.

Example 2.1 Let $X = [0, \infty)$ be endowed with the metric d , with $d(x, y) = |x - y|$. Let $T: X \rightarrow X$ be a single-valued map

$$Tx = \begin{cases} \frac{x^2}{4}, & \text{if } x \in [0, 1] \\ x, & \text{if } x \in (1, \infty) \end{cases}$$

and

$$\alpha(x, y) = \begin{cases} 1, & \text{if } x, y \in [0, 1] \\ 0, & \text{otherwise} \end{cases}$$

If we denote $\psi(t) = \frac{t}{2}$ for all $t \geq 0$ we can check if T is $\alpha - \psi$ -contractive mapping. It is obvious that for all $x, y \in X$, T is an $\alpha - \psi$ -contractive mapping and T is α -admissible. So T has at least one fixed point.

Our next result is related to paper [13]. We introduce a new contraction condition (see paper [7] Theorem 2.3) in order to get a new improvement of the results given there for the case of generalized contraction mappings.

Theorem 2.3 Let (X, d) be a complete metric space and $T: X \rightarrow X$ be a single-valued mapping such that

$$\psi(d(Tx, Ty)) \leq \phi \left(\frac{\alpha d(y, Ty)[1 + d(x, Tx)]}{1 + d(x, y)} + \beta d(x, y) \right)$$

where $\alpha, \beta > 0$ with $\alpha + \beta < 1$ and $\psi, \phi: [0, \infty) \rightarrow [0, \infty)$ are two functions with the following conditions:

i) $\psi(a) \leq \phi(b) \Rightarrow a \leq b$;

ii) T is continuous.

Then T has a fixed point and for any given $x_0 \in X$, the iterative sequence $T^n x_0$ converges to this fixed point.

Proof. Let $x_0 \in X$ and $x_1 = Tx_0$ be arbitrary chosen. If $d(Tx_0, Tx_1) = 0$, then $Tx_0 = Tx_1$. This proves that $x_1 \in F_T$.

Consider $x_n := Tx_{n-1}$ for all n .

Suppose that $d(Tx_0, Tx_1) \neq 0$. Notice that $d(x_1, x_2) = d(Tx_0, Tx_1)$. If $d(x_1, x_2) = 0$ then $x_2 \in F_T$. Suppose that $d(x_1, x_2) \neq 0$. So we have

$$\begin{aligned} \psi(d(x_1, x_2)) &= \psi(d(Tx_0, Tx_1)) \\ &\leq \phi\left(\frac{\alpha d(x_1, Tx_1)[1 + d(x_0, Tx_0)]}{1 + d(x_0, x_1)} + \beta d(x_0, x_1)\right) \\ &\leq \phi\left(\frac{\alpha d(x_1, x_2)[1 + d(x_0, x_1)]}{1 + d(x_0, x_1)} + \beta d(x_0, x_1)\right) \\ &\leq \phi(\alpha d(x_1, x_2) + \beta d(x_0, x_1)). \end{aligned}$$

Using i) we have $d(x_1, x_2) \leq \alpha d(x_1, x_2) + \beta d(x_0, x_1)$.

So $d(x_1, x_2) \leq \frac{\beta}{1-\alpha} d(x_0, x_1)$.

By induction we get that $d(x_{n+1}, x_n) \leq \left(\frac{\beta}{1-\alpha}\right)^n d(x_0, x_1)$.

Notice that $r = \frac{\beta}{1-\alpha} < 1$.

Moreover, for $n < m$, we have

$$\begin{aligned} d(x_n, x_m) &\leq d(x_{n-1}, x_n) + \dots + d(x_{m-1}, x_m) \\ &\leq (r^n + \dots + r^{m-1})d(x_0, x_1) \\ &\leq \frac{r^n}{1-r} d(x_0, x_1) \rightarrow 0 \quad \text{as } n, m \rightarrow \infty. \quad (*) \end{aligned}$$

Therefore $\{x_n\}$ is a Cauchy sequence. Since (X, d) is a complete space, there exists $z \in X$ such that $x_n \rightarrow x^*$ as $n \rightarrow \infty$. From the continuity of T , it follows that $x_{n+1} = Tx_n \rightarrow Tx^*, n \rightarrow \infty$. By uniqueness of the limit, we get $x^* = Tx^*$, that is $x^* \in F_T$. ■

Example 2.2 Let X be endowed with the metric $d: X \times X \rightarrow [0, \infty)$ and $T: X \rightarrow X$ be a single-valued mapping,

$$Tx = \begin{cases} 2, & \text{if } x \in [1, 2] \\ \frac{5}{2}, & \text{if } x \in (2, 3) \\ 1, & \text{otherwise} \end{cases}$$

It is easy to verify that this example verify Theorem 2.3 for $\alpha = \frac{1}{9}, \beta = \frac{7}{8}$. We can choose $\psi(t) = \frac{1}{t}$ and $\phi(t) = t$ with $t \in [1, +\infty)$, so all the conditions will be verified for this values. Then T has many fixed points, such as $2, \frac{5}{2}$.

The next fixed point theorem is an extension of Theorem 2.2 from [12], by using a weakly contractive condition.

Theorem 2.4 Let (X, d) be a complete metric space, $\alpha: X \times X \rightarrow X$ be a function, and $T: X \rightarrow X$ be a single-valued mapping. Suppose that the following conditions take place

- i) T is a generalized α -Geraghty contraction type;
- ii) T is triangular α -admissible;
- iii) There exists $x_0 \in X$ such that $\alpha(x_0, Tx_0) \geq 1$.

Then T has a fixed point.

Proof. Let $x_0 \in X$ such that $\alpha(x_0, Tx_0) \geq 1$. Define $\{x_n\}$ by $x_{n+1} = Tx_n$ for all $n \in \mathbb{N}$. If $x_n = x_{n+1}$, than the proof is finished.

Suppose that $x_n \neq x_{n+1}$. Since T is triangular α -admissible then we have

$$\alpha(x_1, x_2) \geq 1 \quad \text{and} \quad \alpha(x_1, x_3) \geq 1.$$

Due to Remark 1.1 we have

$$\alpha(x_n, x_{n+1}) \geq 1 \quad \text{and} \quad \alpha(x_n, x_{n+2}) \geq 1$$

for all $n \in \mathbb{N}$.

Notice that $\alpha(x_n, x_{n+m}) \geq 1$ for all $m, n \in \mathbb{N}$. Now we have to prove that $\lim_{n \rightarrow \infty} d(x_n, x_{n+1}) = 0$.

Using Definition 1.1 and from condition ii) we get

$$\begin{aligned}
 d(x_1, x_2) &\leq \alpha(x_0, x_1)d(Tx_0, Tx_1) \\
 &\leq \beta(M(x_0, x_1))M(x_0, x_1) \\
 &\leq \beta\left(\max\{d(x_0, x_1), \frac{d(x_1, Tx_1)[1 + d(x_0, Tx_0)]}{1 + d(x_0, x_1)}\}\right) \\
 &\quad \times \max\{d(x_0, x_1), \frac{d(x_1, Tx_1)[1 + d(x_0, Tx_0)]}{1 + d(x_0, x_1)}\} \\
 &\leq \beta(\max\{d(x_0, x_1), d(x_1, x_2)\}) \cdot \max\{d(x_0, x_1), d(x_1, x_2)\}
 \end{aligned}$$

If $\max\{d(x_0, x_1), d(x_1, x_2)\} = d(x_1, x_2)$, then we will have a contradiction because of the definition of β .

So $\max\{d(x_0, x_1), d(x_1, x_2)\} = d(x_0, x_1)$. Thus we conclude

$$d(x_1, x_2) \leq \beta(d(x_0, x_1)) \cdot d(x_0, x_1) < d(x_0, x_1) \quad (1)$$

If $d(Tx_1, Tx_2) = 0$ then $x_2 \in F_T$.

Suppose that $d(Tx_1, Tx_2) \neq 0$, then

$$\begin{aligned}
 d(x_2, x_3) &\leq \alpha(x_1, x_2)d(Tx_1, Tx_2) \\
 &\leq \beta(M(x_1, x_2))M(x_1, x_2) \\
 &\leq \beta\left(\max\{d(x_1, x_2), \frac{d(x_2, Tx_2)[1 + d(x_1, Tx_1)]}{1 + d(x_1, x_2)}\}\right) \\
 &\quad \times \max\{d(x_1, x_2), \frac{d(x_2, Tx_2)[1 + d(x_1, Tx_1)]}{1 + d(x_1, x_2)}\} \\
 &\leq \beta(\max\{d(x_1, x_2), d(x_2, x_3)\}) \cdot \max\{d(x_1, x_2), d(x_2, x_3)\}
 \end{aligned}$$

For case 1 we choose $\max\{d(x_1, x_2), d(x_2, x_3)\} = d(x_2, x_3)$, which is a contradiction.

So $\max\{d(x_1, x_2), d(x_2, x_3)\} = d(x_1, x_2)$. From here we can conclude that

$$d(x_2, x_3) \leq \beta(d(x_1, x_2)) \cdot d(x_1, x_2) < d(x_1, x_2) \quad (2)$$

By induction we have

$$d(x_n, x_{n+1}) < \beta(d(x_{n-1}, x_n)) \cdot d(x_{n-1}, x_n) < d(x_{n-1}, x_n) \quad (3)$$

Therefore the sequence $\{d(x_{n-1}, x_n)\}$ is a positive and nonincreasing sequence. Hence, there exists $r \geq 0$ such that $\lim_{n \rightarrow \infty} d(x_{n-1}, x_n) = r$. Suppose that $r \neq 0$, then we have

$$\frac{d(x_n, x_{n+1})}{d(x_{n-1}, x_n)} \leq \beta(d(x_{n-1}, x_n)) < 1$$

This implies that $\lim_{n \rightarrow \infty} \beta(d(x_{n-1}, x_n)) = 1$. But $\beta \in \mathfrak{F}$ which implies $d(x_{n-1}, x_n) = 0$. Hence this is a contradiction, so $r = 0$.

Suppose that $\{x_n\}$ is not a Cauchy sequence. Then there exists $\varepsilon > 0$ such that, for all $k \geq 1$, there exists $m_k > n_k > k$ with $d(x_{n_k}, x_{m_k-1}) \geq \varepsilon$. Let m_k be the smallest number satisfying the conditions above. Hence, we have $d(x_{n_k}, x_{m_k-1}) < \varepsilon$. Therefore, we get

$$\varepsilon \leq d(x_{n_k}, x_{m_k-1}) + d(x_{m_k-1}, x_{m_k}) < \varepsilon + d(x_{m_k-1}, x_{m_k}).$$

Letting $k \rightarrow \infty$, we get $\lim_{k \rightarrow \infty} d(x_{n_k}, x_{m_k}) = \varepsilon$. Since

$$|d(x_{n_k}, x_{m_k-1}) - d(x_{n_k}, x_{m_k})| \leq d(x_{m_k}, x_{m_k-1})$$

we obtain $\lim_{k \rightarrow \infty} d(x_{n_k}, x_{m_k-1}) = \varepsilon$.

By the same procedure we obtain

$$\lim_{k \rightarrow \infty} d(x_{m_k}, x_{n_k-1}) = \lim_{k \rightarrow \infty} d(x_{m_k-1}, x_{n_k-1}) = \varepsilon.$$

Using Remark 1.1 we get $d(x_{m_k-1}, x_{n_k-1}) \geq 1$. This yields to

$$\begin{aligned} d(x_{n_k}, x_{m_k}) &= d(Tx_{n_k-1}, Tx_{m_k-1}) \\ &\leq \alpha(x_{n_k-1}, x_{m_k-1})d(Tx_{n_k-1}, Tx_{m_k-1}) \\ &\leq \beta(M(x_{n_k-1}, x_{m_k-1})) \cdot M(x_{n_k-1}, x_{m_k-1}) \end{aligned}$$

where

$$\begin{aligned} M(x_{n_k-1}, x_{m_k-1}) &= \\ &= \max\{d(x_{n_k-1}, x_{m_k-1}), \frac{d(x_{m_k-1}, Tx_{m_k-1})[1 + d(x_{n_k-1}, Tx_{n_k-1})]}{1 + d(x_{n_k-1}, x_{m_k-1})}\} \\ &= \max\{d(x_{n_k-1}, x_{m_k-1}), d(x_{m_k-1}, x_{m_k})\}. \end{aligned}$$

We can see that $\lim_{k \rightarrow \infty} M(x_{n_k-1}, x_{m_k-1}) = \varepsilon$.

Hence, we get

$$\frac{d(x_{n_k}, x_{m_k})}{M(x_{n_k-1}, x_{m_k-1})} \leq \beta(M(x_{n_k-1}, x_{m_k-1})).$$

Letting $k \rightarrow \infty$, we get $\lim_{k \rightarrow \infty} \beta(M(x_{n_k-1}, x_{m_k-1})) = 1$. By the definition of $\beta \in \mathfrak{F}$ we have $\lim_{k \rightarrow \infty} M(x_{n_k-1}, x_{m_k-1}) = 0$. Hence, $\varepsilon = 0$, which is a contradiction.

Therefore $\{x_n\}$ is a Cauchy sequence. Since (X, d) is a complete metric space we get that $\lim_{n \rightarrow \infty} x_n = x^*$.

So we have

$$\begin{aligned} d(x_{n+1}, Tx^*) &\leq d(Tx_n, Tx^*) \\ &\leq \alpha(x_n, x^*)d(Tx_n, Tx^*) \\ &< \beta(M(x_n, x^*)) \cdot M(x_n, x^*) \quad (4) \end{aligned}$$

We know that

$$\begin{aligned} M(x_n, x^*) &= \max\left\{d(x_n, x^*), \frac{d(x^*, Tx^*)[1 + d(x_n, Tx_n)]}{1 + d(x_n, x^*)}\right\} \\ &= \max\left\{d(x_n, x^*), \frac{d(x^*, Tx^*)[1 + d(x_n, x_{n+1})]}{1 + d(x_n, x^*)}\right\} \end{aligned}$$

But $x_{n+1} = Tx_n$ and $\lim_{n \rightarrow \infty} x_n = x^*$, then we get

$$M(x_n, x^*) = d(x^*, Tx^*).$$

Introducing this in (4) then we will have

$$d(x_{n+1}, Tx^*) \leq \beta(d(x^*, Tx^*)) \cdot d(x^*, Tx^*).$$

Suppose that $d(x^*, Tx^*) > 0$. From $M(x_n, x^*) = d(x^*, Tx^*)$ we get that $M(x_n, x^*) > 0$, which implies $\beta(M(x_n, x^*)) < M(x_n, x^*)$.

Letting $x_n \rightarrow x^*$, then we will have $d(x^*, Tx^*) < d(x^*, Tx^*)$, which is a contradiction. So we obtain $d(x^*, Tx^*) = 0$, which proves that $x^* \in F_T$. ■

References

- [1] I. Cabrera, J. Harjani, K. Sadarangani, A fixed point theorem for contractions of rational type in partially ordered metric spaces, *Annali dell'Universita di Ferrara, Sez. VII Sci. Mat.*, **59**(2013), No.2 , 251-258.
- [2] M. Frigon, A. Granas, Résultats du type de Leray-Schauder pour les contractions multivoques, *Topol. Methods Nonlinear Anal.*, **4**(1994), 197-208.
- [3] Geraghty, On contractive mappings , *Proc. Am. Math. Soc.*, **40**(1973), 604-608.
- [4] J. Hasanzade, S. Rezapour and N. Shahzad, On fixed points $\alpha - \psi$ -contractive multifunctions, *Fixed Point Theory and Applications*, **1**(2012).

- [5] N. Hussain, M. A. Kutbi, and P. Salimi, Fixed Point Theory in α -Complete Metric Spaces with Applications , *Abstract and Applied Analysis*, **Art. ID 280817**(2014), 11 pages.
- [6] E. Karapinar, B. Samet , Generalized $\alpha - \psi$ contractive type mappings and related fixed point theorems with applications, *Abstr. Appl. Anal.*, **Art. ID 793486**(2012).
- [7] Madjid Eshaghi Gordji, Hamid Baghani and Gwang Hui Kim , Common fixed point theorems for (ψ, ϕ) -weak nonlinear contraction in partially ordered sets , *Fixed Point Theory and Applications*, **62**(2012).
- [8] A. Oprea , Fixed point theorems for multivalued generalized contractions of rational type in complete metric spaces, *Creative Mathematics and Informatics*, **23**(2014), No. 1, 99-106.
- [9] Ovidiu Popescu , Some new fixed point theorems for α -Geraghty contraction type maps in metric spaces, *Fixed Point Theory and Applications*, (2014).
- [10] A. Petruşel, I.A. Rus , The theory of a metric fixed point theorem for multivalued operators, *Yokohama Publ., Yokohama*, (2010), 161-175.
- [11] B. Samet, Fixed points for $\alpha - \psi$ -contractive mappings with an application to quadratic integral equations , *Electronic Journal of Differential Equations*, **2014**(2014), No.152, 1-18.
- [12] B. Samet, C. Vetro, P. Vetro, Fixed point theorem for $\alpha - \psi$ - contractive type mappings, *Nonlinear Anal.*, **75**(2012), 2154-2165.
- [13] B. Samet and H. Yazidi, An extension of Banach fixed point theorem for mappings satisfying a contractive condition of integral type, *Int. J. Math. Anal.*, **3**(2009), No.26, 1265-1271.
- [14] Seong-Hoon Cho, Jong-Sook Bae and Erdal Karapınar, Fixed point theorems for α -Geraghty contraction type maps in metric spaces , *Fixed Point Theory and Applications*, (2013).
- [15] Yongfu Su and Jen-Chih Yao, Further generalized contraction mapping principle and best proximity theorem in metric spaces, to appear.