

Aesthetic shapes generated by piecewise continuous Bézier curves

FLORINA RUSU IOANA CRĂCIUN LUCIAN TUDOSE
(CLUJ-NAPOCA) (CLUJ-NAPOCA) (CLUJ-NAPOCA)

ABSTRACT. The aesthetic criteria have been intensely studied since Antiquity due to the importance of aesthetics in everyday life. In general, decision-makers try to make the optimal decisions with respect to several criteria including aesthetic ones. In this paper we propose an aesthetic optimization model of a basic shape generated by piecewise continuous Bézier curves using evolutionary algorithms. The objective function of the optimization is formulated using the Harmonic Ratio as an aesthetic standard.

KEY WORDS: aesthetic standards, mathematical mean, Harmonic Ratio, optimization, evolutionary algorithms

MSC 2010: 93B40; 68W20

◇Florina Rusu, Technical University of Cluj-Napoca,
Department of Mechanical System Engineering,
email: Florina.Rusu@omt.utcluj.ro

◇Ioana Crăciun, Technical University of Cluj-Napoca,
Department of Mechanical System Engineering,
email: Ioana.Craciun@omt.utcluj.ro

◇Lucian Tudose, Technical University of Cluj-Napoca,
Department of Mechanical System Engineering,
email: Lucian.Tudose@omt.utcluj.ro

1 Introduction

Since ancient times the problem of establishing aesthetic standards in nature, art, science, and engineering has been addressed by scientists. The purpose was and is to find mathematical models that allow quantifying the aesthetic value brought by the beauty and harmony of the aesthetic objects. Several theories of art or aesthetics such as mystical, pedagogic, and hedonistic have been developed during the years. In opposition to these theories have appeared the analytic ones that are focused on identifying the most frequently encountered aesthetic factors and the general laws that govern them [1].

Many researchers have focused on defining aesthetic standards using the Golden Ratio or the generalizations of the Golden Ratio (see for example [2], [3], [4], and [5]).

Recall that a segment $[AB]$ is divided by a point $C \in [AB]$ in Golden Ratio if

$$(1.1) \quad \frac{AC}{CB} = \frac{CB}{AB} \quad \text{or} \quad CB = \sqrt{AC \cdot AB}$$

i.e. CB is the geometric mean of AC and AB . Taking account of this remark in [6] the authors introduced a notion of generalized Golden Ratio defined by an arbitrary mathematical homogeneous mean $M(x, y)$, $x, y > 0$. We recall that definition. Suppose that the equation

$$(1.2) \quad t = M(1, 1 + t)$$

admits a unique solution in $(1, \infty)$.

Let $[AB]$ be a segment in $\mathbb{R}^2$, $C \in [AB]$, $CA = x$, $CB = y$, $x < y$. We say that C divides $[AB]$ in M -Golden Ratio if

$$(1.3) \quad y = M(x, x + y)$$

The ratio $\frac{y}{x} = \varphi_M$ is called in this case M -Golden Ratio. If M is the geometric mean we obtain the classical Golden Ratio. If M is the harmonic



Figure 1: Golden Ratio

mean, i.e.

$$(1.4) \quad M(x, y) = \frac{2xy}{x + y}, \quad x, y > 0$$

we get $\varphi_M = \sqrt{2}$ and we call it the Harmonic Ratio. See for more details [6, 7]. Hereinafter we will denote the Harmonic Ratio by $\varphi_H = \sqrt{2}$.

The Golden Ratio was used as an aesthetic standard in many branches of science and art since Antiquity. Some authors consider that the importance of the Golden Ratio in nature, as well as in art, is exaggerated (see for example [8, 9]). In this paper we will consider the Harmonic Ratio φ_H as an aesthetic standard. An argument in favor of the Harmonic Ratio is given in [8] and consists in the analysis of some natural aesthetic forms. As mentioned in [8] the nautilus, an octopus in a shell that consists of a series of chambers, is definitely not in the shape of a spiral based on the golden ratio. Moreover, the conducted measurements prove that the ratios ranged from 1.24 to 1.43 and the average was 1.33. A spiral based on φ_H is considerably closer to the shape of the nautilus than the one based on the Golden Ratio.

2 Quadratic Bézier curves

A lot of fields in the computational science domain, such as computer aided geometric design, require describing complex objects. The description of such objects is usually obtained using basic elements (points, curves, surfaces) which are handled with the help of mathematical functions such as polynomials and rational functions. Bézier curves are one of the main curves used in 3D modeling. They have been intensely exploited and several techniques have been developed based on them. Moreover, these techniques found industrial use in automobile design [10].

Our approach is based on Bézier curves. Recall first some notions related to quadratic Bézier curves.

Let $P_0(x_0, y_0)$, $P_1(x_1, y_1)$, $P_2(x_2, y_2)$ three points in $\mathbb{R}^2$ and denote $\vec{b}_0 = \vec{OP_0}$, $\vec{b}_1 = \vec{OP_1}$, $\vec{b}_2 = \vec{OP_2}$. Then the quadratic Bézier curve determined by P_0, P_1, P_2 is generated by the function $\vec{B} : [0, 1] \rightarrow \mathbb{R}^2$

$$(2.1) \quad \vec{B}(t) = (1-t)^2 \cdot \vec{b}_0 + 2t(1-t) \cdot \vec{b}_1 + t^2 \cdot \vec{b}_2, \quad t \in [0, 1]$$

If $M(x, y)$ is an arbitrary point on the Bézier curve then

$$(2.2) \quad \begin{cases} x = (1-t)^2 \cdot x_0 + 2t(1-t) \cdot x_1 + t^2 \cdot x_2 \\ y = (1-t)^2 \cdot y_0 + 2t(1-t) \cdot y_1 + t^2 \cdot y_2 \end{cases}$$

The slopes of the tangent line to the Bézier curve at P_0 and P_2 are given by:

$$(2.3) \quad \begin{cases} \vec{B}'(0) = 2(\vec{b}_1 - \vec{b}_0) \\ \vec{B}'(1) = 2(\vec{b}_2 - \vec{b}_1) \end{cases}$$

3 Mathematical model

3.1 Basic shape generated using Bézier curves

Let there be seven points $A(0, y_1)$, $M(u, 0)$, $P(v, y_2)$, $B(x_2, y_2)$, $Q(w, 0)$, $C(x_3, y_3)$, and N and the three Bézier curves determined by them as shown in Figure 2. The coordinates of the points have to meet the following conditions:

$$(3.1) \quad \begin{cases} 0 < y_1 < y_2 \\ 0 < y_3 < y_2 \\ 0 < u < v < x_2 < w < x_3 \end{cases}$$

The point N is chosen so that the two curve arcs $\widehat{AN}$ and $\widehat{NP}$ are smoothly connected. Therefore $N(su + (1-s)v, (1-s)y_2)$ with $s \in (0, 1)$

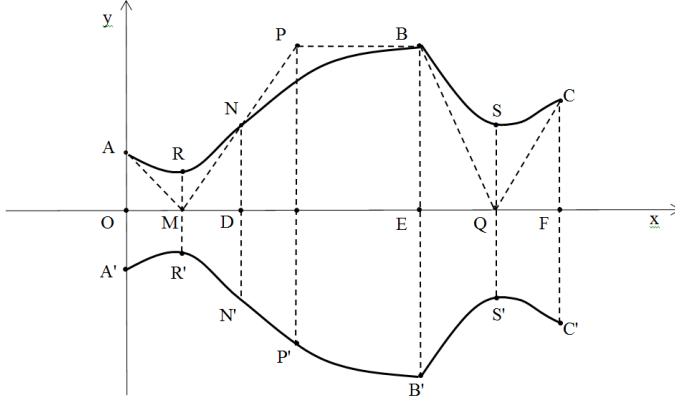


Figure 2: The vertical section of a vase generated by three Bézier curves

and

$$(3.2) \quad \widehat{AN} : \begin{cases} x = 2t(1-t)u + t^2(su + (1-s)v) \\ y = (1-t)^2y_1 + t^2(1-s)y_2 \end{cases}, t \in [0, 1]$$

$$(3.3) \quad \widehat{NB} : \begin{cases} x = (1-t)^2(su + (1-s)v) + 2t(1-t)v + t^2x_2 \\ y = (1-t)^2(1-s)y_2 + 2t(1-t)y_2 + t^2y_2 \end{cases}, t \in [0, 1]$$

The tangency condition in N is:

$$(3.4) \quad \begin{cases} su + (1-s)v = \frac{u+v}{2} \\ (1-s)y_2 = \frac{y_2}{2} \end{cases}$$

which leads to $N(\frac{u+v}{2}, \frac{y_2}{2})$. The third Bézier curve arc is given by:

$$(3.5) \quad \widehat{BC} : \begin{cases} x = (1-t)^2x_2 + 2t(1-t)w + t^2x_3 \\ y = (1-t)^2y_2 + t^2y_3 \end{cases}, t \in [0, 1]$$

3.2 Problem formulation

The general purpose of the optimization conducted in this paper is to obtain a vase made out of three Bézier curves that is aesthetically optimal. In order to do this several qualitative aspects have been defined. The optimization encompasses six aesthetic standards. The first considered aesthetic standard refers to the height-width ratio of the vase which is desired to be the Harmonic Ratio:

$$(3.6) \quad \frac{OF}{BB'} = \varphi_H$$

which can be rewritten in the following form:

$$(3.7) \quad \frac{x_3}{2y_2} = \varphi_H$$

The second qualitative aspect that can be imposed is the ratio between the lengths of the segments ND and RM to be equal to the Harmonic Ratio:

$$(3.8) \quad \frac{y_2}{2y_1(1-t_0)^2 + t_0^2 y_2} = \varphi_H$$

where t_0 is the positive subunitary solution of the equation:

$$(3.9) \quad t^2 \left(\frac{u+v}{2} - 2u \right) + 2tu - u = 0$$

The third aesthetic standard defined for the vase is the condition that the opening-width ratio has to be equal to the Harmonic Ratio:

$$(3.10) \quad \frac{BE}{CF} = \varphi_H \iff \frac{y_2}{y_3} = \varphi_H$$

The fourth aesthetic standard refers to the base of the vase:

$$(3.11) \quad \frac{AO}{ND} = \varphi_H \iff \frac{2y_1}{y_2} = \varphi_H$$

The last two aesthetic standards considered in the optimization are based

on different ratios of the lengths of the segments that compose the height of the vase in order to ensure an aesthetically optimal vase:

$$(3.12) \quad \frac{DE}{OD} = \varphi_H \iff \frac{2x_2 - u - v}{u + v} = \varphi_H$$

$$(3.13) \quad \frac{EQ}{QF} = \varphi_H \iff \frac{w - x_2}{x_3 - w} = \varphi_H$$

4 Aesthetic optimization

4.1 Optimization using evolutionary algorithms

The main advantage of the evolutionary algorithms consists in the fact that they mimic the best elements from nature, especially from the biological systems that have evolved over so many years through natural selection. One of these algorithms is Cuckoo Search developed in 2009 by Xin-She Yang and Suash Deb [11] and based on the so-called Lévy flights. This algorithm was used to find the optimal aesthetic shape generated by piecewise continuous Bézier curves.

Cuckoo Search based on Lévy flights was implemented in MATLAB R2009b and can be described using the following three rules [11]:

1. Each cuckoo lays one egg at a time, and dumps it in a randomly chosen nest using Lévy flights;
2. The best nests with high-quality eggs will be carried over to the next generations;
3. The number of available host nests is fixed, and the egg laid by a cuckoo is discovered by the host bird with a probability $pa \in (0, 1)$. In this case, the host bird can either get rid of the egg, or simply abandon the nest and build a completely new nest.

The program begin with the random generation of an initial population of n cuckoos within the limits of the search space representing the possible configurations of vases. A cuckoo is a vector of eight variables whose values

are necessary for drawing the vase using three Bézier curves (see 3.2). For every generation each possible vase configuration is evaluated using the objective function and the best one is found.

The cuckoos migrate towards the area with the best aesthetic shapes with respect to the chosen aesthetic criteria. This migration is implemented in a repetitive structure and it is based on the three already mentioned rules. It has to be mentioned that each new cuckoo chick which was obtained by laying an egg in a nest replaces its parent in the population only if it is better than its parent.

The migration ends when the maximum allowed number of evaluations of the objective function $Nmax$ is reached and the optimal vases with respect to the chosen aesthetic criteria are returned as solution.

4.2 Objective function

The optimization proposed in this paper focuses on achieving the above mentioned aesthetic standards. The objective function that is minimized is given by:

$$(4.1) \quad f(y_1, u, v, y_2, x_2, w, x_3, y_3) = f_1(y_1, u, v, y_2, x_2, w, x_3, y_3) + \lambda \cdot f_2(y_1, u, v, y_2, x_2, w, x_3, y_3)$$

where

$$(4.2) \quad \begin{aligned} f_1(y_1, u, v, y_2, x_2, w, x_3, y_3) = & \left(\frac{x_3}{2y_2} - \varphi_H \right)^2 + \left(\frac{y_2}{y_3} - \varphi_H \right)^2 \\ & + \left(\frac{y_2}{2y_1(1-t_0)^2 + t_0^2 y_2} - \varphi_H \right)^2 \\ & + \left(\frac{2y_1}{y_2} - \varphi_H \right)^2 + \left(\frac{2x_2 - u - v}{u + v} - \varphi_H \right)^2 \\ & + \left(\frac{w - x_2}{x_3 - w} - \varphi_H \right)^2 \end{aligned}$$

Table 1: Optimization parameters

Parameter	Significance	Value
λ	Penalty factor	10^{10}
n	Number of cuckoos	25
$Nmax$	The maximum allowed number of objective function evaluations	700,000
pa	The discovery probability	0.25

$$\begin{aligned}
f_2(y_1, u, v, y_2, x_2, w, x_3, y_3) = & \max \{0, y_1 - y_2 + \epsilon\} + \max \{0, y_3 - y_2 + \epsilon\} \\
& + \max \{0, v - x_2 + \epsilon\} + \max \{0, x_2 - x_3 + \epsilon\} \\
(4.3) \quad & + \max \{0, u - v + \epsilon\} + \max \{0, x_2 - w + \epsilon\} \\
& + \max \{0, w - x_3 + \epsilon\}
\end{aligned}$$

$\epsilon = 10^{-3}$ and t_0 is the positive subunitary solution of the equation (3.9). The symbol λ is used for the penalty factor that directs the search of the optimal solution towards the shapes that are composed out of Bézier curves that do not overlap.

4.3 Optimization results

The optimization problem was formulated as a minimization problem. The search domain for the variables y_1 , y_2 , and y_3 is $[0, 3]$ and for the variables u , v , x_2 , w , and x_3 is $[0, 9]$. The parameters required for the optimization are given in Table 1.

The optimization based on the six aesthetic standards defined using the Harmonic Ratio finds an infinity of aesthetic shapes generated using Bézier curves that are aesthetically optimal. Their shape varies from vases with a tall narrow foot and a large middle part as it can be seen in Figure 3a to vases with quite a narrow small foot and a large opening as shown in Figure 3b. Moreover, vases similar to the well-known Greek amphora are

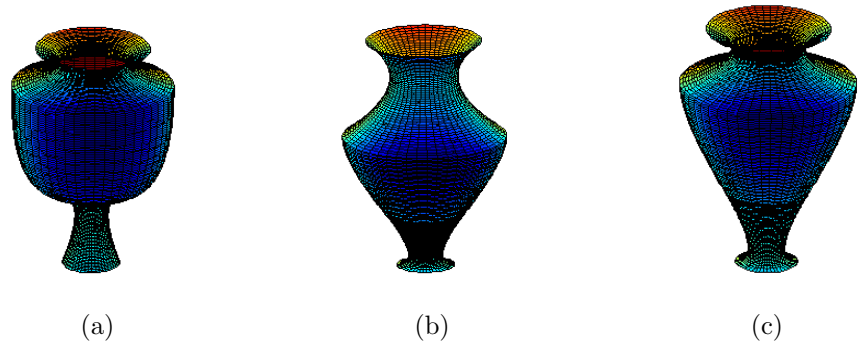


Figure 3: The results of the aesthetical optimization

obtained as optimal solutions (see Figure 3b and 3c). So, the obtained shapes correspond to classical aesthetic forms defined in Antiquity (see for more details [1]). However, all the obtained vases are optimal with respect to the six aesthetic standards defined in section 3.2. Choosing just one vase from the infinity of optimal vases can be made using the human factor.

5 Conclusions

The Golden Ratio, defined by a proportion corresponding to the geometric mean, has been used to define aesthetic standards by many artists and scientists. However, the Harmonic Ratio, defined by a proportion corresponding to the harmonic mean, has been less exploited regardless of the fact that it also occurs in nature. Using the Harmonic Ratio the problem of obtaining aesthetically optimal basic shapes is formulated in this paper as a minimization problem of eight variables and then solved. The proposed mathematical model allows formulating easily other aesthetic standards for the given shape using the Harmonic Ratio or other sacred geometry constant.

The optimization model can be adapted without difficulty for a different shape or for using more aesthetic criteria. Actually if one wants to obtain a single optimal vase with respect only to the aesthetic criteria and not let the human factor decide, then other aesthetic standards have to be formulated.

Acknowledgement

This paper is supported by the Sectoral Operational Programme Human Resources Development POSDRU/159/1.5/S/137516 financed from the European Social Fund and by the Romanian Government. This paper was also supported by the RKB Group, the Swiss bearing manufacturer and the authors greatly appreciate and wish to thank all the RKB staff for their great assistance during this project.

References

- [1] G. Birkhoff. *Aesthetic Measure*, Cambridge: Harvard University Press, 1933.
- [2] R.A. Dunlop. *The Golden Ratio and Fibonacci Numbers*, World Scientific, 1997.
- [3] M. Livio. *The Golden Ratio: The Story of Phi, the Worlds's Most Astonishing Number*, Broadway Books, 2002.
- [4] A. P. Stakhov. The Generalized Principle of the Golden Section and its applications in mathematics, science, and engineering, *Chaos, Solitons and Fractals*, **26** (2005), 263–289.
- [5] A. P. Stakhov. The generalized golden proportions, a new theory of real numbers, and ternary mirror-symmetrical arithmetic, *Chaos, Solitons and Fractals*, **33** (2007), 315–334.
- [6] I. Crăciun, D. Inoan, D. Popa and L. Tudose. Generalized Golden Ratios defined by means, *Appl. Math. Comput.*, **250** (2015), 221–227.
- [7] I. Crăciun and L. Tudose. Golden Ratio-an aesthetic standard in nature, science and engineering, *Proceedings of the 2nd Conference for PhD students in Civil Engineering*, Dec. 2014, 683–689.
- [8] C. Falbo. The Golden Ratio-A Contrary Viewpoint, *The College Mathematics Journal*, **36** (2005), 123–134.
- [9] G. Markowsky. Misconceptions about the golden ratio, *The College Mathematics Journal*, **23** (1992), 2–19.
- [10] G.E. Farin, J. Hoschek and M.S. Kim. *Handbook of Computer Aided Geometric Design*, Elsevier, 2002.
- [11] X.-S. Yang and S. Deb. Cuckoo search via Lévy flights, *NaBIC, IEEE*, Dec. 2009, 210–214.