

Fixed set theorems in the multivalued setting

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ABSTRACT. Using the properties of the Pompeiu-Hausdorff metric, we will prove some fixed point and fixed set theorems for multivalued type operators.

KEY WORDS: Pompeiu-Hausdorff metric, fixed point and fixed set theorems.

MSC 2010: 47H10, 47H04.

1 Introduction and Preliminaries

In this paper we will prove some fixed point and fixed set theorems for multivalued type operators.

We will present first some notions and symbols used in this paper.

Let (X, d) be a metric space. Throughout this paper we will use the following notations and concepts:

$$P(X) = \{Y \subseteq X \mid Y \neq \emptyset\};$$

$$P_{cp}(X) = \{Y \in P(X) \mid Y \text{ is nonempty compact}\};$$

$$P_{cv}(X) = \{Y \in P(X) \mid Y \text{ is nonempty convex}\};$$

$$P_{b,cl}(X) = \{Y \in P(X) \mid \overline{Y} = Y, \text{diam}(Y) < \infty\};$$

$$P_{cp,cv}(X) = P_{cp}(X) \cap P_{cv}(X).$$

We begin with the definition:

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- (1) D is called the gap functional between A and B : $D : P(X) \times P(X) \rightarrow R_+$, $D(A, B) = \inf\{d(a, b) \mid a \in A, b \in B\}$.
In particular, if $x_0 \in X$ then $D(x_0, B) := D(\{x_0\}, B)$.
- (2) $diam : P(X) \rightarrow R_+ \cup \{+\infty\}$, $diam A := \sup\{d(a, b) \mid a, b \in A\}$.
- (3) $\rho : P(X) \times P(X) \rightarrow R_+ \cup \{+\infty\}$, $\rho(A, B) = \sup\{D(a, B) \mid a \in A\}$.
- (4) ρ is called the (generalized) excess functional.
- (5) $H : P(X) \times P(X) \rightarrow R_+ \cup \{+\infty\}$, $H(A, B) = \max\{\rho(A, B), \rho(B, A)\}$,
 H is the (generalized) Pompeiu-Hausdorff functional on $P(X)$.

Theorem 1.1 *Let (X, d) be a metric space, then $(P_{b,cl}(X), H)$ is a metric space.*

Some important properties of Pompeiu- Hausdorff functional.

Lemma 1.1 *Let (X, d) be a metric space and $Y, Z, V, W \in P(X)$. Then we have:*

- (i) $H(Y, Z) = 0$ if and only if $\overline{Y} = \overline{Z}$;
- (ii) $H(Y, Z) = H(\overline{Y}, \overline{Z})$;
- (iii) $H(Y \cup V, Z \cup W) \leq \max\{H(Y, Z), H(V, W)\}$;
- (iv) Let $Y, Z \in P(X)$ and $\varepsilon > 0$. Then for each $y \in Y$ there exists $z \in Z$ such that $d(y, z) \leq H(Y, Z) + \varepsilon$;
- (v) Let $Y, Z \in P(X)$ and $q > 1$. Then for each $y \in Y$ there exists $z \in Z$ such that $d(y, z) \leq q \cdot H(Y, Z)$;
- (vi) If $Y, Z \in P_{cp}(X)$ then for each $y \in Y$ there exists $z \in Z$ such that $d(y, z) \leq H(Y, Z)$;
- (vii) If $Y, Z \in P(X)$ then $H(Y, Z) \leq \varepsilon$ is equivalent with the following assertion: for each $y \in Y$ there exists $z \in Z$ such that $d(y, z) \leq \varepsilon$ and for each $z \in Z$ there exists $y \in Y$ with $d(y, z) \leq \varepsilon$;

(viii) Let $\varepsilon > 0$. If $Y, Z \in P(X)$ are such that $H(Y, Z) < \varepsilon$ then for each $y \in Y$ there exists $z \in Z$ such that $d(y, z) < \varepsilon$.

Let A, B be two families of nonempty subsets of X and let $P(B)$ be the family of all nonempty subsets of B .

Definition 1.1 Let $\phi : A \rightarrow P(B)$ such that there exists $F \in \phi(A)$ satisfying a relation of the type $A \subset F, A \supset F, A \cap F \neq \emptyset$, is called a semifixed set of multivalued ϕ . Moreover, a fixed set for ϕ is any set $Z \in A$ satisfying $Z \in \phi(Z)$.

Let X be a Banach space and denote for now on $K := P_{cp}(P_{cp}(X))$.

The space K is endowed with the Pompeiu-Hausdorff distance H induced by the metric h of $P_{cp}(X)$, i.e $H(A, B) := \max\{e(A, B), e(B, A)\}$,

$$\text{where } e(A, B) = \sup_{A \in A} \inf_{B \in B} h(A, B).$$

For $A, A', B, B' \in P_{cp}(X)$ and $\lambda \in R$, the set:

$D(A, B) := \inf\{\|a - b\| \mid a \in A, b \in B\}$, we have:

- (i) $D(A, B) = D(B, A)$;
- (ii) $D(A, B) = 0$ if only if $A \cap B \neq \emptyset$;
- (iii) $D(\lambda A, \lambda B) = |\lambda|D(A, B)$;
- (iv) $D(A, B) \leq D(A', B') + h(A, A') + h(B, B')$;
- (v) $h(A, B) \leq \text{diam}(A) + \text{diam}(B) + D(A, B)$.

The function D is continuous on $P_{cp}(X) \times P_{cp}(X)$,

$$\left| \sup_{B \in B} D(A, B) - \sup_{B \in B} D(A', B) \right| \leq h(A, A').$$

We define:

$$\Delta(A, B) := \max\{f(B, A), f(A, B)\}, \text{ for } A, B \in K, \text{ where}$$

$$f(A, B) = \inf_{A \in A} \sup_{B \in B} D(A, B) \text{ and } f(B, A) = \inf_{B \in B} \sup_{A \in A} D(B, A).$$

Definition 1.2 A function $\varphi : R_+ \rightarrow R_+$ is a comparison function if it satisfies: φ is monotone increasing and $(\varphi^n(t))_{n \in \mathbb{N}}$ converges to 0, for all $t > 0$.

Remark 1.1 If $\varphi : R_+ \rightarrow R_+$ is a comparison function then $\varphi(0) = 0$ and $\varphi(t) < t$, for every $t > 0$.

Definition 1.3 A map $\phi : A \rightarrow K$ is said compact if its range $\phi(A) = \{Y \in P_{cp}(X) | Y \in \phi(X) \text{ for some } X \in A\}$ is precompact in $P_{cp}(X)$.

As $P_{cp}(X)$ is complete, ϕ is compact if and only if $\phi(A)$ has compact closure in $P_{cp}(X)$.

Definition 1.4 Let X be a Banach space, $\mathcal{A}$ be a subset of $P_{cp}(P_{cp}(X))$. Then $\phi : \mathcal{A} \rightarrow P_{cp}(P_{cp}(X))$ is said to be a set φ -contraction if $\varphi : R_+ \rightarrow R_+$ is a comparison function and

$$\Delta(\phi(Y), \phi(Z)) \leq \varphi(D(Y, Z)), \text{ for all } Y, Z \in \mathcal{A}.$$

Definition 1.5 Let $(X, \mathcal{H})$ be a metric space. The multivalued operator $\phi : X \rightarrow P_{cp}(P_{cp}(X))$, is called upper semicontinuous (resp. lower semicontinuous) if, for every $x_0 \in X$ and $\varepsilon > 0$ there exists an open neighborhood V of x_0 such that $\mathcal{H}(\phi(x), \phi(x_0)) < \varepsilon$ (resp. $\mathcal{H}(\phi(x_0), \phi(x)) < \varepsilon$), for every $x \in V$. ϕ is called continuous if it is both, upper and lower semicontinuous.

The first result is:

Theorem 1.2 (I.C. Tise [5]) Let X be a Banach space, $\mathcal{A}$ be a closed subset of $P_{cp}(P_{cp}(X))$ and let $\phi : \mathcal{A} \rightarrow P_{cp}(P_{cp}(X))$ be a strong compact and upper semicontinuous multivalued, with values $\phi(X) \subset \mathcal{A}$ for every $X \in \mathcal{A}$. Suppose there exists a comparison function $\varphi : R_+ \rightarrow R_+$ such that satisfying the following condition:

$$\Delta(\phi(Y), \phi(Z)) \leq \varphi(D(Y, Z)) \text{ for every } Y, Z \in \mathcal{A}.$$

Then there exists $A \in \mathcal{A}$ and $F \in \phi(A)$ such that:

$$A \cap F \neq \emptyset.$$

In the papers [5], [6], we have studied existence, uniqueness and data dependence for the fixed points, of operators $\phi : A \rightarrow K$. In this article we will discuss further aspects of the fixed point and fixed set theory.

2 Main Results

Let X be a Banach space. The space $P_{cp}(P_{cp}(X))$ endowed with the Pompeiu-Hausdorff metric $\mathcal{H}$ induced by the metric H of $P_{cp}(X)$,

$$\mathcal{H}(\mathcal{A}, \mathcal{B}) := \max\{e(\mathcal{A}, \mathcal{B}), e(\mathcal{B}, \mathcal{A})\},$$

where $e(\mathcal{A}, \mathcal{B}) := \sup_{A \in \mathcal{A}} \inf_{B \in \mathcal{B}} H(A, B)$, and $e(\mathcal{B}, \mathcal{A}) := \sup_{B \in \mathcal{B}} \inf_{A \in \mathcal{A}} H(B, A)$.

Remark 2.1 $P_{cp,cv}(P_{cp,cv}(X)) \subset P_{cp}(P_{cp}(X))$.

Definition 2.1 Let (X, d) be a metric space and $\tilde{T} : X \rightarrow P_{cp}(P_{cp}(X))$ be an operator. Then $\tilde{T}$ is said to be:

(i) α - set contraction if $\alpha \in [0, 1)$ and

$$\mathcal{H}(\tilde{T}(x_1), \tilde{T}(x_2)) \leq \alpha d(x_1, x_2), \text{ for all } x_1, x_2 \in X.$$

(ii) (ϵ, α) -contraction if $\epsilon > 0$, $\alpha \in [0, 1)$ and

$$\mathcal{H}(\tilde{T}(x_1), \tilde{T}(x_2)) \leq \alpha d(x_1, x_2), \text{ for all } x_1, x_2 \in X \text{ with } d(x_1, x_2) < \epsilon.$$

We have the following result for $\tilde{Y}, \tilde{Z} \in P_{cp}(P_{cp}(X))$.

Lemma 2.1 Let (X, d) be a complete metric space and $\tilde{Y}, \tilde{Z} \in P_{cp}(P_{cp}(X))$ and $q > 1$. Then for all $Y \in \tilde{Y}$ exists $Z \in \tilde{Z}$ such that

$$H(Y, Z) \leq q\mathcal{H}(\tilde{Y}, \tilde{Z}).$$

The following result is an extension of the Nadler's Contraction Principle.

Theorem 2.1 *Let (X, d) be a complete metric space and $\tilde{T} : X \rightarrow P_{cp}(P_{cp}(X))$ be an α - set contraction. Then, there exists $x^* \in X$ such that $\{x^*\} \in \tilde{T}(x^*)$.*

Proof Let $x_0 \in X$, $x_1 \in X_1$ and $X_1 \in \tilde{T}(x_0)$.

We have for $X_1 \in \tilde{T}(x_0)$ exists $X_2 \in \tilde{T}(x_1)$ such that:

$$(2.1) \quad H(X_1, X_2) \leq q\mathcal{H}(\tilde{T}(x_0), \tilde{T}(x_1)), \text{ where } q > 1.$$

Also, for X_1, X_2 and $x_1 \in X_1$ exists $x_2 \in X_2$ such that:

$$(2.2) \quad d(x_1, x_2) \leq qH(X_1, X_2), \text{ where } q > 1.$$

Then by (2.1), (2.2), Lemma 1.1(v), Lemma 2.1 and $\tilde{T} - \alpha$ set contraction condition we have:

$$d(x_1, x_2) \leq qH(X_1, X_2) \leq q^2\mathcal{H}(\tilde{T}(x_0), \tilde{T}(x_1)) \leq q^2\alpha d(x_0, x_1).$$

We choose $1 < q < \frac{1}{\sqrt{\alpha}}$, thus we can construct the sequence $(x_n)_{n \in N}$ given by $x_n \in X_n$ with $X_n \in \tilde{T}(x_{n-1})$ such that:

$$d(x_n, x_{n+1}) \leq (q^2\alpha)^n d(x_0, x_1).$$

Then

$$\begin{aligned} d(x_n, x_{n+p}) &\leq \alpha d(x_n, x_{n+1}) + \cdots + d(x_{n+p-1}, x_{n+p}) \\ &\leq ((q^2\alpha)^n + \cdots + (q^2\alpha)^{n+p-1}) d(x_0, x_1) \\ &\leq \frac{(q^2\alpha)^n}{1 - q^2\alpha} d(x_0, x_1) \rightarrow 0 \text{ as } n, p \rightarrow \infty. \end{aligned}$$

Then, the sequence $(x_n)_{n \in N}$ converges to $x^* \in X$. Since $\tilde{T}$ has closed graph and $x_n \in \tilde{T}(x_{n-1})$, for all $n \in N$ we get that $\{x^*\} \in \tilde{T}(x^*)$.

The proof is complete.

Another result is a data dependence theorem.

Theorem 2.2 *Let (X, d) be a complete metric space and $\tilde{T}, \tilde{S} : X \rightarrow P_{cp}(P_{cp}(X))$ be two α -set contractions. If there exists $\eta > 0$ such that*

$$\mathcal{H}(\tilde{T}(x), \tilde{S}(x)) \leq \eta, \text{ for all } x \in X,$$

then we have $\mathcal{H}(F_{\tilde{T}}, F_{\tilde{S}}) \leq \frac{\alpha\eta}{1-\alpha}$.

Proof By Theorem 2.1 we know that: $F_{\tilde{T}} \neq \emptyset$ and $F_{\tilde{S}} \neq \emptyset$.

Let $x_T^* \in F_{\tilde{T}}$ and for $x_T^* \in \tilde{T}(x_T^*)$, since

$$\mathcal{H}(\tilde{T}(x), \tilde{S}(x)) \leq \eta, \text{ for all } x \in X,$$

there exists $Y_1 \in \tilde{S}(x_T^*)$ such that:

$$\begin{aligned} H(\{x_T^*\}, Y_1) &\leq q\mathcal{H}(\tilde{T}(x_T^*), \tilde{S}(x_T^*)) \\ &\leq q\eta, \text{ for any } q > 1. \end{aligned}$$

Now for Y_1 and $x_T^* \in \tilde{T}(x_T^*)$ there exists $y_1 \in Y_1$ such that:

$$d(x_T^*, y_1) \leq qH(\{x_T^*\}, Y_1).$$

Thus

$$(2.3) \quad d(x_T^*, y_1) \leq q^2\eta.$$

As in the proof of Theorem 2.1 we can construct a sequence $(y_n)_{n \in N}$ with

$$\begin{cases} y_0 = x_T^* \\ y_n \in Y_n \text{ with } Y_n \in S(y_{n-1}), & n \in N^* \end{cases}$$

which converges to an element $x_S^* \in F_{\tilde{S}}$.

Moreover

$$d(y_n, y_{n+p}) \leq (q^2\alpha)^n \frac{1 - (q^2\alpha)^p}{1 - q^2\alpha} d(y_0, y_1), \text{ for all } n, p \in N.$$

As $p \rightarrow \infty \implies d(y_n, x_S^*) \leq (q^2\alpha)^n \frac{1}{1 - q^2\alpha} d(y_0, y_1)$, for all $n \in N$.

Then, for $n = 0$ we get

$$d(x_T^*, x_S^*) \leq \frac{1}{1 - q^2\alpha} d(y_0, y_1).$$

Since $y_0 = x_T^* \in \tilde{T}(x_T^*)$, $y_1 \in S(x_T^*)$ and the relation (2.3) we get that:

$$d(x_T^*, x_S^*) \leq \frac{q^2\eta}{1 - q^2\alpha}.$$

By, this relation and the analogous one obtained by interchanging the roles of x_T^* and x_S^* we get that

$$\mathcal{H}(F_{\tilde{T}}, F_{\tilde{S}}) \leq \frac{q^2\eta}{1 - q^2\alpha}.$$

Let get $q \searrow 1$ we obtain $\mathcal{H}(F_{\tilde{T}}, F_{\tilde{S}}) \leq \frac{\eta}{1 - q}$.

The proof is complete.

A result for Ulam-Hyers stability is:

Theorem 2.3 *Let (X, d) be a complete metric space and $\tilde{T} : X \rightarrow P_{cp}(P_{cp}(X))$ be an α - set contraction. Let $\epsilon > 0$ and $x \in X$, $X \in \tilde{T}(x)$ such that $\mathcal{H}(x, \tilde{T}(x)) \leq \epsilon$. Then there exists $x^* \in F_{\tilde{T}}$ such that $d(x, x^*) \leq \frac{\epsilon}{1 - \alpha}$.*

Proof Let $x \in X, \epsilon > 0$ are such that $\mathcal{H}(x, \tilde{T}(x)) < \epsilon$.

Since $\tilde{T}(x)$ is compact and there exists $y \in \tilde{T}(x)$ such that $d(x, y) \leq \epsilon$.

By the proof of Theorem 2.1, then we have that:

$$d(x, x^*) \leq \frac{1}{1 - \alpha} d(x, y).$$

We get the conclusion $d(x, x^*) \leq \frac{\epsilon}{1-\alpha}$.

The proof is complete.

We denote: $(SF)_{\tilde{T}} := \{x \in Y | x = \tilde{T}(x)\}$ the strict fixed point set of $\tilde{T}$ and by $Graph(\tilde{T}) := \{(x, y) \in X \times Y : y \in \tilde{T}(x)\}$ the graphic of $\tilde{T}$.

We discuss near the well-posedness of the fixed point problem.

Theorem 2.4 *Let (X, d) be a complete metric space and $\tilde{T} : X \rightarrow P_{cp}(P_{cp}(X))$ be an α - set contraction, with $(SF)_{\tilde{T}} \neq \emptyset$. If $(x_n)_{n \in \mathbb{N}}$ is a sequence in X such that $\mathcal{H}(x_n, \tilde{T}(x_n)) \rightarrow 0$ as $n \rightarrow \infty$, then $(x_n) \rightarrow x^*$ as $n \rightarrow \infty$.*

Proof Let $(x_n)_{n \in \mathbb{N}}$ be a sequence in X such that $\mathcal{H}(x_n, \tilde{T}(x_n)) \rightarrow 0$ as $n \rightarrow \infty$. Then:

$$\begin{aligned} d(x_n, x^*) &\leq \mathcal{H}(x_n, \tilde{T}(x_n)) + \mathcal{H}(\tilde{T}(x_n), \tilde{T}(x^*)) \\ &\leq \mathcal{H}(x_n, \tilde{T}(x_n)) + \alpha d(x_n, x^*). \end{aligned}$$

Then $d(x_n, x^*) \leq \frac{1}{1-\alpha} \mathcal{H}(x_n, \tilde{T}(x_n))$ as $n \rightarrow +\infty$.

The proof is complete.

Next result presents the limit shadowing property with respect to $\tilde{T}$.

Theorem 2.5 *Let (X, d) be a complete metric space. If $(y_n)_{n \in \mathbb{N}}$ is a sequence in X such that $\mathcal{H}(y_{n+1}, \tilde{T}(y_n)) \rightarrow 0$ as $n \rightarrow \infty$, then there exists a sequence $(x_n)_{n \in \mathbb{N}} \subset X$ of successive approximations for $\tilde{T}$, such that $d(x_n, y_n) \rightarrow 0$ as $n \rightarrow \infty$.*

Proof Let $(y_n)_{n \in \mathbb{N}}$ be a sequence in X such that $\mathcal{H}(y_{n+1}, \tilde{T}(y_n)) \rightarrow 0$ as $n \rightarrow \infty$. Then, there exists $u_n \in \tilde{T}(y_n)$, $n \in \mathbb{N}$ such that $d(y_{n+1}, u_n) \rightarrow 0$ as $n \rightarrow +\infty$.

We shall prove that $d(y_n, x^*) \rightarrow 0$ as $n \rightarrow +\infty$. We have:

$$\begin{aligned}
d(x^*, y_{n+1}) &\leq \mathcal{H}(x^*, \tilde{T}(y_n)) + \mathcal{H}(y_{n+1}, \tilde{T}(y_n)) \\
&= \mathcal{H}(\tilde{T}(x^*), \tilde{T}(y_n)) + \mathcal{H}(y_{n+1}, \tilde{T}(y_n)) \\
&\leq \alpha d(x^*, y_n) + \mathcal{H}(y_{n+1}, \tilde{T}(y_n)) \\
&\leq \alpha[\alpha d(x^*, y_{n-1}) + \mathcal{H}(y_n, \tilde{T}(y_{n-1}))] + \mathcal{H}(y_{n+1}, \tilde{T}(y_n)) \\
&\leq \dots \leq \alpha^{n+1} d(x^*, y_0) + \alpha^n \mathcal{H}(y_1, \tilde{T}(y_0)) + \dots + \mathcal{H}(y_{n+1}, \tilde{T}(y_n)).
\end{aligned}$$

Thus, by Cauchy's Lemma, we have $d(x^*, y_{n+1}) \rightarrow 0$ as $n \rightarrow +\infty$ (see[3]). On the other hand, by the proof of Theorem 2.4, we know that there exists a sequence $(x_n)_{n \in \mathbb{N}}$ of successive approximations for $\tilde{T}$ starting from arbitrary $(x_0, x_1) \in \text{Graph}(\tilde{T})$ which converge to a fixed point $x^* \in X$ of the operator $\tilde{T}$. Since, the fixed point is unique, we get that $d(x_n, x^*) \rightarrow 0$ as $n \rightarrow +\infty$. Hence, for such a sequence $(x_n)_{n \in \mathbb{N}}$, we have

$$d(y_n, x_n) \leq d(y_n, x^*) + d(x^*, x_n) \rightarrow 0 \text{ as } n \rightarrow +\infty.$$

The proof is complete.

References

- [1] V. Lakshmikantham, T. Gnana Bhaskar, J. Vasundhara Devi. *Theory of Set Differential Equations in Metric Spaces*. Cambridge Scientific Publishers, 2006.
- [2] S.B. Nadler. Multivalued contraction mappings, *Pacific J. Math.*, **30** (1969), 475–488.
- [3] A. Petruşel, I.A. Rus. The theory of a metric fixed point theorem for multivalued operators, *Proc. of the Ninth Intern. Conf. on Fixed Point Theory and its Applications, Changhua, Taiwan, July 16-22, 2009, Yokohama Publ.* (2011), 161–175.
- [4] A. Petruşel, I.A. Rus, M.A. Şerban. Fixed points, fixed sets and iterated multifunction systems for nonself multivalued operators, *Set-Valued Var. Anal.*, **23** (2015), 223–237.

- [5] I.C. Tișe. *Equations in Spaces of Multivalued Functions*, Ph.D. Dissertation, Babeș-Bolyai University Cluj-Napoca, 2010.
- [6] I.C. Tișe. A fixed point approach for functional-integral set equations, *Demonstratio Mathematica*, **43**(4) (2010), 815–826.