

Explicit representation of the approximation of the solutions of some diffusion equations

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ABSTRACT. In this article we approximate the solutions of the Cauchy problems associated with some diffusion equations. We use some results from the theory of semigroups of operators. We get the approximation of the solution in explicit form and we give an estimation of the approximation error. In some cases we obtain the exact solutions of the Cauchy problems. Some examples are given.

KEY WORDS: diffusion equations, Cauchy problems, semigroup of operators, Bernstein type operators, estimation of the approximation error

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1 Introduction

Consider the following initial boundary value problem

$$(1.1) \quad \begin{cases} v_t = k(x(1-x)v_{xx} + (a+1-(a+b+2)x)v_x), & 0 \leq x \leq 1, t \geq 0, \\ v(0, x) = f(x), & x \in [0, 1], \end{cases}$$

where $a, b, k \in \mathbb{R}$, $k > 0$.

The problem (1.1) is related to some mathematical models arising from biology, genetics, population dynamics or mathematical finance (see [1], [2], [3], [4], [5], [6], [7], [8], [14]).

Using the theory of the semigroups of operators, the exact solution of the problem (1.1) can be obtained as limit of the iterates of some positive linear operators.

It is difficult to calculate the limit for an arbitrary function f . We approximate the function f by a polynomial P_m of a given degree m and we consider a second initial boundary value problem

$$(1.2) \quad \begin{cases} v_t = k(x(1-x)v_{xx} + (a+1-(a+b+2)x)v_x), & x \in [0, 1], t \geq 0, \\ v(0, x) = P_m(x), & x \in [0, 1], \end{cases}$$

where $a, b, k \in \mathbb{R}$, $k > 0$. Using some formulas from [10] (see also [11]), we determine the exact analytical solution v_m of the problem.

The solution v of the problem (1.1) will be approximated by the solution v_m of the problem (1.2).

In Section 2 we present some connections between C_0 -semigroups and Cauchy problems. More about this topic can be found in the monograph [2]. In Section 3 we present the main result. We use semigroups associated with some Bernstein type operators to get the solutions of some Cauchy problems of the form (1.1) and (1.2). Some examples are given in Section 4.

2 C_0 -semigroups and Cauchy problems

Let X be a Banach space over $\mathbb{R}$.

A family $(T(t))_{t \geq 0}$ of bounded linear operators on X is called a semigroup if $T(0) = I$ and $T(t+s) = T(s) \circ T(t)$ for every $s, t \geq 0$.

The sequence $(T(t))_{t \geq 0}$ is said to be strongly continuous or a C_0 -semigroup if

$$\lim_{t \rightarrow t_0} T(t)f = T(t_0)f$$

for every $t_0 \geq 0$ and $f \in X$.

More about the semigroups of operators can be found in [2].

Let $(T(t))_{t \geq 0}$ be a C_0 -semigroup of operators on the space X , and let $A : D(A) \rightarrow X$, $D(A) \subset X$, be its infinitesimal generator. The solution $u : [0, \infty) \rightarrow X$ of the Cauchy problem

$$\begin{cases} u'(t) = A(u(t)) & , t \geq 0 \\ u(0) = f & , f \in D(A) \end{cases}$$

is given by $u(t) = T(t)f$, $t \geq 0$.

The solution $v : [0, \infty) \rightarrow X$ of the problem

$$\begin{cases} v'(t) = kA(v(t)) & , t \geq 0 \\ v(0) = f & , f \in D(A) \end{cases}$$

with $k > 0$ is given by $v(t) = T(kt)f$, $t \geq 0$.

We take $X = C[0, 1]$ with the supremum norm.

For the differential operator

$$(2.1) \quad Au(x) = x(1-x)u''(x) + (a+1-(a+b+2)x)u'(x).$$

we set

$$(2.2) \quad D(A) = \begin{cases} D_M(A), & a \geq 0, b \geq 0 \\ D_{VM}(A), & a = -1, b \geq 0 \\ D_{MV}(A), & a \geq 0, b = -1 \end{cases}$$

where

$$D_M(A) = \{u \in C[0, 1] \cap C^2(0, 1) : Au \in C[0, 1]\}$$

$$D_{VM}(A) = \{u \in C[0, 1] \cap C^2(0, 1) : \lim_{x \rightarrow 0} Au(x) = 0, \lim_{x \rightarrow 1} Au(x) \in \mathbb{R}\}$$

$$D_{MV}(A) = \{u \in C[0, 1] \cap C^2(0, 1) : \lim_{x \rightarrow 1} Au(x) = 0, \lim_{x \rightarrow 0} Au(x) \in \mathbb{R}\}.$$

The Cauchy problem for the diffusion equation associated to the

differential operator A given by (2.1) is

$$(2.3) \quad \begin{cases} u_t = x(1-x)u_{xx} + (a+1-(a+b+2)x)u_x, & 0 \leq x \leq 1, \ t \geq 0 \\ u(0, x) = f(x), & x \in [0, 1]. \end{cases}$$

Let $(T(t))_{t \geq 0}$ be a C_0 -semigroup associated with (2.3), i.e. $u(t, x) = T(t)f(x)$. Then the C_0 -semigroup associated with (1.1) is $(T(kt))_{t \geq 0}$, i.e. $v(t, x) = T(kt)f(x)$.

In particular $v(t, x) = u(kt, x)$.

If we consider the differential operator

$$(2.4) \quad Au(x) = \frac{x(1-x)}{2}u''(x)$$

with the domain

$$(2.5) \quad D(A) = \{u \in C[0, 1] \cap C^2(0, 1) : \lim_{x \rightarrow 0} Au(x) = \lim_{x \rightarrow 1} Au(x) = 0\}$$

then the associated Cauchy problem is

$$(2.6) \quad \begin{cases} u_t = \frac{x(1-x)}{2}u_{xx}, & 0 \leq x \leq 1, \ t \geq 0 \\ u(0, x) = f(x), & x \in [0, 1]. \end{cases}$$

Let $(T(t))_{t \geq 0}$ be a C_0 -semigroup associated with (2.6), i.e. $u(t, x) = T(t)f(x)$. It follows that the C_0 -semigroup associated with (1.1) for $a = b = -1$ is $(T(2kt))_{t \geq 0}$, i.e. $v(t, x) = T(2kt)f(x)$.

In next section we approximate the solutions of the Cauchy problems (2.3) and (2.6).

3 Semigroups associated to some Bernstein type operators

Let $a, b \geq -1$ be real numbers.

In [9], the authors considered the following modified Bernstein operator

$L_n : C[0, 1] \rightarrow C[0, 1]$, $n \geq \max\{a + 1, b + 1\}$ given by

$$(3.1) \quad L_n f = \sum_{k=0}^n f \left(\left(1 - \frac{a+b+2}{2n} \right) \frac{k}{n} + \frac{a+1}{2n} \right) p_{n,k}, \quad f \in C[0, 1]$$

where

$$p_{n,k}(x) = \binom{n}{k} x^k (1-x)^{n-k}, \quad x \in [0, 1].$$

For $a = b = -1$ we get the classical Bernstein operator

$$(3.2) \quad B_n f = \sum_{k=0}^n f \left(\frac{k}{n} \right) p_{n,k}, \quad f \in C[0, 1].$$

We have the following result from [3].

Theorem 3.1 *Let A be the differential operator from (2.1) with the domain $D(A)$ given by (2.2). Then $(A, D(A))$ is the infinitesimal generator of a positive contractive C_0 -semigroup $(T(t))_{t \geq 0}$ on $C[0, 1]$. For every $f \in C[0, 1]$ and $t \geq 0$,*

$$T(t)f = \lim_{n \rightarrow \infty} L_n^{k(n)} f, \quad \text{uniformly on } [0, 1]$$

where $(k(n))_{n \geq 1}$ is an arbitrary sequence of positive integers such that $k(n)/n \rightarrow 2t$ and $L_n^{k(n)}$ is the iterate of L_n of order $k(n)$.

The exact solution of the Cauchy problem (2.3) is $u(t, x) = T(t)f(x)$, $x \in [0, 1]$, $t \geq 0$.

We approximate f by a polynomial of degree m

$$f(x) \approx P_m(x) = \sum_{j=0}^m a_j x^j.$$

and we consider the Cauchy problem

$$(3.3) \quad \begin{cases} u_t = x(1-x)u_{xx} + (a+1 - (a+b+2)x)u_x, & 0 \leq x \leq 1, \quad t \geq 0 \\ u(0, x) = P_m(x), & x \in [0, 1]. \end{cases}$$

The exact solution of the Cauchy problem (3.3) is $u_m(t, x) = T(t)P_m(x)$,

$x \in [0, 1]$, $t \geq 0$. We have

$$(3.4) \quad u_m(t, x) = \lim_{n \rightarrow \infty} L_n^{k(n)} \left(\sum_{j=0}^m a_j e_j \right) (x) = \sum_{j=0}^m a_j \lim_{n \rightarrow \infty} L_n^{k(n)} e_j(x) \\ = \sum_{j=0}^m a_j T(t) e_j(x).$$

The images of the monomials under $T(t)$ are given by the next theorem (see [10]).

Theorem 3.2 *Let $(y)_0 = 1$, $(y)_l = y(y-1)\dots(y-l+1)$ for $l \in \mathbb{R}$, $l \geq 1$. Then we have*

$$(3.5) \quad T(t)e_r = \sum_{i=0}^r c_{i,r}(t)e_i, \quad r \geq 0, \quad t \geq 0,$$

where

$$c_{i,r}(t) = \sum_{j=i}^r \alpha_{ijk} \exp(-j(j+a+b+1)t), \quad 0 \leq i \leq r$$

and

$$\alpha_{ijr} = (-1)^{i+j} \binom{r}{i} \binom{r-i}{j-i} (r+a)_{r-i} \frac{2j+a+b+1}{(r+j+a+b+1)_{r-i+1}}, \quad 0 \leq i \leq j \leq r.$$

Next we consider the Cauchy problem (2.6).

From [1] (see also [2]) we have

Theorem 3.3 *Let A be the differential operator from (2.4) with the domain $D(A)$ given by (2.5). Then $(A, D(A))$ is the infinitesimal generator of a positive contractive C_0 -semigroup $(T(t))_{t \geq 0}$ on $C[0, 1]$. For every $f \in C[0, 1]$ and $t \geq 0$,*

$$T(t)f = \lim_{n \rightarrow \infty} B_n^{k(n)} f, \quad \text{uniformly on } [0, 1]$$

where $(k(n))_{n \geq 1}$ is an arbitrary sequence of positive integers such that $k(n)/n \rightarrow t$ and $B_n^{k(n)}$ is the iterate of B_n of order $k(n)$.

The exact solution of the Cauchy problem (2.6) is $u(t, x) = T(t)f(x)$, $x \in [0, 1]$, $t \geq 0$.

We approximate the function f by a polynomial P_m of degree m and consider the Cauchy problem

$$(3.6) \quad \begin{cases} u_t = \frac{x(1-x)}{2} u_{xx}, & 0 \leq x \leq 1, \quad t \geq 0 \\ u(0, x) = P_m(x), & x \in [0, 1]. \end{cases}$$

Then $u_m(x, t) = T(t)P_m(x)$ is the exact solution of the Cauchy problem (3.6). We have

$$(3.7) \quad \begin{aligned} u_m(t, x) &= \lim_{n \rightarrow \infty} B_n^{k(n)} \left(\sum_{j=0}^m a_j e_j \right) (x) = \sum_{j=0}^m a_j \lim_{n \rightarrow \infty} B_n^{k(n)} e_j(x) \\ &= \sum_{j=0}^m a_j T(t) e_j(x). \end{aligned}$$

We can calculate $T(t)e_j$ using the formulas from Theorem 3.2 with $a = b = -1$, $t := t/2$ and taking the index of summation in (3.5) from 1 to r , or from the paper [11] of Kelisky and Rivlin.

Remark 3.1 *If $a, b \geq 0$, $k = 1$ or $a = b = -1$, $k = 1/2$ we can use the Mache operator (see [9]) instead of the modified Bernstein operator (3.1). The results are given in [12] and [13].*

The following theorem gives the estimation for the approximation error of the function u by the function u_m .

Theorem 3.4 *We have the following estimation*

$$(3.8) \quad |u(t, x) - u_m(t, x)| \leq \|f - P_m\|, \quad x \in [0, 1], \quad t \geq 0.$$

Proof. Indeed

$$\begin{aligned} |u(t, x) - u_m(t, x)| &= |T(t)f(x) - T(t)P_m(x)| \\ &\leq \|T(t)f - T(t)P_m\| = \|T(t)(f - P_m)\| \\ &\leq \|T(t)\| \|f - P_m\|. \end{aligned}$$

Taking into account that $\|T(t)\| = 1$ we get the estimation (3.8). ■

From Theorem 3.4 it follows that the approximation error of the solution of the Cauchy problems (2.3) and (2.6) (and therefore of the Cauchy problem (1.1)) depends only of the approximation error of the function f by the polynomial P_m . If the function f is a polynomial then we can get the exact solution of the Cauchy problems.

If $f \in C^{m+1}[0, 1]$, then we approximate f by the Taylor polynomial of degree m

$$f(x) \approx T_m f(x) = \sum_{j=0}^m \frac{(x - \frac{1}{2})^j}{j!} f^{(j)}\left(\frac{1}{2}\right), \quad x \in [0, 1].$$

We can write

$$T_m f(x) = \sum_{j=0}^m \frac{(T_m f)^{(j)}(0)}{j!} x^j.$$

It follows that

$$a_j = \frac{(T_m f)^{(j)}(0)}{j!}, \quad j = 0, 1, \dots, m.$$

The approximation error is

$$(3.9) \quad f(x) - T_m f(x) = \frac{(x - \frac{1}{2})^{m+1}}{(m+1)!} f^{(m+1)}(c), \quad c \in (0, 1), \quad x \in [0, 1].$$

Using (3.4) and (3.7) respectively, we get the approximation of the solutions of the Cauchy problems (2.3) and (2.6) respectively,

$$u_m(t, x) = \sum_{j=0}^m \frac{(T_m f)^{(j)}(0)}{j!} T(t) e_j(x), \quad x \in [0, 1], \quad t \geq 0,$$

where $T(t) e_j$ is given by Theorem 3.2. From (3.8) and (3.9), we get the following estimation

$$(3.10) \quad |u(t, x) - u_m(t, x)| \leq \frac{1}{2^{m+1}(m+1)!} \|f^{(m+1)}\|, \quad x \in [0, 1], \quad t \geq 0.$$

Instead of the Taylor polynomial T_m we can use the Bernstein polynomial

B_m given by (3.2). For every $f \in C[0, 1]$, it has the following representation using monomial basis

$$B_m(f)(x) = \sum_{j=0}^m \frac{j!}{m^j} \binom{m}{j} \left[0, \frac{1}{m}, \dots, \frac{j}{m}; f \right] x^j, \quad x \in [0, 1],$$

where $\left[0, \frac{1}{m}, \dots, \frac{j}{m}; f \right]$ represents the divided differences of the function f on the nodes $0, \frac{1}{m}, \dots, \frac{j}{m}$. It follows that

$$a_j = \frac{j!}{m^j} \binom{m}{j} \left[0, \frac{1}{m}, \dots, \frac{j}{m}; f \right], \quad j = 0, 1, \dots, m.$$

We get the following representation for the approximation of the solution of the Cauchy problems (2.3) and (2.6)

$$u_m(t, x) = \sum_{j=0}^m \frac{j!}{m^j} \binom{m}{j} \left[0, \frac{1}{m}, \dots, \frac{j}{m}; f \right] T(t) e_j(x), \quad x \in [0, 1], \quad t \geq 0,$$

where $T(t)e_j$ is given by Theorem 3.2.

The Bernstein polynomial $B_m f$ preserves the shape of the function f . Using the shape preserving properties of the operator L_n from (3.1) we get that, for every fixed $t \geq 0$, the approximate solution $u(t, \cdot)$ inherits the shape properties of the function f .

The sequence $(B_m f)_{m \geq 1}$ converges to f , for every $f \in C[0, 1]$. If $f \in C^2[0, 1]$ then the error is

$$(3.11) \quad f(x) - B_m(f)(x) = -\frac{x(1-x)}{2m} f''(c), \quad c \in (0, 1), \quad x \in [0, 1].$$

From (3.8) and (3.11) we get the following estimation of the approximation error of the solution of the Cauchy problems (2.3) and (2.6)

$$(3.12) \quad |u(t, x) - u_m(t, x)| \leq \frac{1}{8m} \|f''\|, \quad x \in [0, 1], \quad t \geq 0.$$

If we compare the error estimations (3.10) and (3.12) we observe that in general the error is smaller in the case of using the Taylor polynomial.

But using the Bernstein operator the function f needn't to be of high order differentiable and the approximation solution of the Cauchy problem preserves the shape of the function f .

4 Examples

We consider the initial boundary value problem (1.1). We approximate the solution of this problem for $f = e^x$ and $f = \sin(x - 1/2)$. The function f is approximated by Taylor polynomials and Bernstein polynomials. The maximal approximation errors of the solution of the Cauchy problems are given in the following tables in both cases.

The maximal error when f is approximated by Taylor polynomials.

m	$f = e^x$	$f = \sin(x - 1/2)$
1	0.67957	$2.5 \cdot 10^{-1}$
2	$1.13262 \cdot 10^{-1}$	$4.16667 \cdot 10^{-2}$
3	$1.41577 \cdot 10^{-2}$	$5.20833 \cdot 10^{-3}$
4	$7.07886 \cdot 10^{-4}$	$5.20833 \cdot 10^{-4}$
8	$1.46306 \cdot 10^{-8}$	$1.07646 \cdot 10^{-8}$
12	$5.32874 \cdot 10^{-14}$	$3.92066 \cdot 10^{-14}$
16	$5.83064 \cdot 10^{-20}$	$4.28994 \cdot 10^{-20}$

The maximal error when f is approximated by Bernstein polynomials.

m	$f = e^x$	$f = \sin(x - 1/2)$
1	0.33979	0.12500
2	0.16989	0.06250
3	0.11326	0.04167
4	0.08495	0.03125
8	0.04247	0.01563
12	0.02832	0.01042
16	0.02124	0.00781

From the tables we can see that the errors are smaller when we approximate the function f by the Taylor polynomials. In this case we

get, for not very large values of m , the approximations of the solutions with high accuracy.

Next, we compute approximations of the solutions of the Cauchy problem (1.1) for two different choices of a , b and k . We use the Taylor polynomial to approximate the function f .

For $a = 2$, $b = 3$ and $k = 1$ the approximations of the solution for $m = 1, 2, 3, 4$ are:

i) if $f = e^x$ then we get

$$v_1(t, x) = \left(\frac{13}{14} + e^{-7t} \left(x - \frac{3}{7} \right) \right) \sqrt{e}$$

$$v_2(t, x) = \left(\frac{53}{56} + e^{-7t} \left(\frac{17x}{18} - \frac{17}{42} \right) + e^{-16t} \left(\frac{x^2}{2} - \frac{4x}{9} + \frac{1}{12} \right) \right) \sqrt{e}$$

$$v_3(t, x) = \left(\frac{953}{1008} + e^{-7t} \left(\frac{23x}{24} - \frac{23}{56} \right) + e^{-16t} \left(\frac{21x^2}{44} - \frac{14x}{33} + \frac{7}{88} \right) + e^{-27t} \left(\frac{x^3}{6} - \frac{5x^2}{22} + \frac{x}{11} - \frac{1}{99} \right) \right) \sqrt{e}$$

$$v_4(t, x) = \left(\frac{7625}{8064} + e^{-7t} \left(\frac{1517x}{1584} - \frac{1517}{3696} \right) + e^{-16t} \left(\frac{85x^2}{176} - \frac{85x}{198} + \frac{85}{1056} \right) + e^{-27t} \left(\frac{25x^3}{156} - \frac{125x^2}{572} + \frac{25x}{286} - \frac{25}{2574} \right) + e^{-40t} \left(\frac{x^4}{24} - \frac{x^3}{13} + \frac{5x^2}{104} - \frac{5x}{429} + \frac{1}{1144} \right) \right) \sqrt{e}$$

ii) if $f = \sin(x - 1/2)$ then we get

$$v_1(t, x) = v_2(t, x) = -\frac{1}{14} + e^{-7t} \left(-\frac{3}{7} + x \right)$$

$$v_3(t, x) = v_4(t, x) = -\frac{72}{1008} + e^{-7t} \left(\frac{71x}{72} - \frac{71}{168} \right) + e^{-16t} \left(\frac{x^2}{44} - \frac{2x}{99} + \frac{1}{264} \right) + e^{-27t} \left(-\frac{x^3}{6} + \frac{5x^2}{22} - \frac{x}{11} + \frac{1}{99} \right).$$

For $a = b = -1$ and $k = 1/2$ we get

i) if $f = e^x$ then the approximations are:

$$v_1(t, x) = x\sqrt{e}$$

$$v_2(t, x) = \left(x + \frac{1}{2}e^{-t}x(x-1) \right) \sqrt{e}$$

$$v_3(t, x) = \left(\frac{25}{24}x + \frac{1}{2}e^{-t}x(x-1) + \frac{1}{12}e^{-3t}x(2x^2 - 3x + 1) \right) \sqrt{e}$$

$$v_4(t, x) = \left(\frac{25}{24}x + \frac{41}{80}e^{-t}x(x-1) + \frac{1}{12}e^{-3t}x(2x^2 - 3x + 1) + \frac{1}{120}e^{-6t}x(5x^3 - 10x^2 + 6x - 1) \right) \sqrt{e}$$

ii) if $f = \sin(x - 1/2)$ then the approximations are:

$$v_1(t, x) = v_2(t, x) = x$$

$$v_3(t, x) = v_4(t, x) = \frac{23}{24}x + \frac{1}{12}e^{-3t}x(-2x^2 + 3x - 1).$$

5 Conclusions

Using the method presented in this paper we can approximate the solution of the initial boundary value problem (1.1), for some choices of the parameters a , b and k , with a prescribed error. The accuracy of the approximated solution depends on the accuracy of the approximation of the function f by the polynomial P_m . If f is a polynomial then we get the exact solution of the problem (1.1) for the corresponding choices of the parameters a , b and k .

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