

The number of sums of a set

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ABSTRACT. For two finite sets of real numbers A and B the set of Minkowski sums are $A + B = \{a + b \mid a \in A, b \in B\}$ and if $A = B$ we denote $S(A) = A + A$ [2]. In this paper we determine the possible values which $|S(A)|$ can take for given $|A|$. In particular we show that the minimal value of $|S(A)|$ is $2|A| - 1$ and it is obtained when A is an arithmetic progression [1], [3].

KEY WORDS: Sum of sets, arithmetic progression, geometric progression

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1 The minimum values and maximum values of $|S(A)|$

Let A be a finite set of real numbers which have $n = |A|$ elements. A first evaluation of the numbers of $S(A)$ is:

Proposition 1.1 *For any finite set $A \subset \mathbb{R}$ the following inequalities are verified:*

$$2|A| - 1 \leq |S(A)| \leq \frac{|A| \cdot (|A| + 1)}{2}.$$

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Proof. If $A = \{a_1, a_2, \dots, a_n\}$ and $a_1 < a_2 < \dots < a_n$ then in $S(A)$ the elements:

$$\begin{aligned} a_1 + a_1 &= S_2, \quad a_1 + a_2 = S_3, \dots \\ a_1 + a_n &= S_{n+1}, \quad a_n + a_2 = S_{n+2}, \\ a_n + a_3 &= S_{n+3}, \dots, \\ a_n + a_{n-1} &= S_{2n-1} \text{ and } a_n + a_n = S_{2n} \end{aligned}$$

are distinct, so $|S(A)| \geq 2n - 1 = 2|A| - 1$.

The maximal numbers of distinct sums is the number of ordered pairs (i, j) with $i \leq j$, which is

$$\binom{n}{2} + n = \frac{n(n-1)}{2} + n = \frac{n(n+1)}{2},$$

$$\text{so } |S(A)| \leq \frac{|A| \cdot (|A| + 1)}{2}.$$

□

In the next proposition we show that the maximum of $|S(A)|$ is obtained when A is geometric progression.

Proposition 1.2 *If A is geometric progression of positive numbers, then*

$$|S(A)| = \frac{|A| \cdot (|A| + 1)}{2}.$$

Proof. Let $q > 0$, $q \neq 1$ be the ratio of progression $A = \{a_1, a_2, \dots, a_n\}$. Thus $a_1 > 0$, $a_2 = a_1q$, $a_3 = a_1q^2, \dots$, $a_n = a_1q^{n-1}$. We will show that all sums $a_i + a_j$ and $a_k + a_l$ are distinct for all pairs $\{i, j\} \neq \{k, l\}$. If by contradiction we assume $a_i + a_j = a_k + a_l$, then $q^{i-1} + q^{j-1} = q^{k-1} + q^{l-1}$ and we can assume $i < k < l < j$. Using the graphic of the function $f(x) = q^x$, $x \geq 0$, which is convex for $q > 1$ and concave for $q < 1$, it follows that $q^{i-1} + q^{j-1} < q^{k-1} + q^{l-1}$ if $q \in (0, 1)$ and $q^{i-1} + q^{j-1} > q^{k-1} + q^{l-1}$, if $q > 1$. □

Corollary 1.1 *If $A = \{1, 2, 2^2, \dots, 2^{n-1}\}$ then*

$$|S(A)| = \frac{n(n+1)}{2}.$$

The next proposition give an example for which is realized the minimum of $|S(A)|$.

Proposition 1.3 *If $A = \{1, 2, \dots, n\}$, then*

$$|S(A)| = 2n - 1 = 2|A| - 1.$$

Proof. $S(A) = \{2, 3, 4, \dots, 2n\}$. □

2 The characterization of arithmetic progression through the map $|S|$

We will show that the arithmetic progressions are the only sets A which realizes the minimum of the function $|S(A)|$.

Theorem 2.1 *The real numbers $a_1 < a_2 < \dots < a_n$ are in arithmetic progression if and only if*

$$|S(\{a_1, a_2, \dots, a_n\})| = 2n - 1.$$

Proof. If $A = \{a_1, a_2, \dots, a_n\}$ are in arithmetic progression, then

$$\begin{aligned} S(A) &= \{2a_1 + (i + j - 2)r \mid i, j = 1, 2, \dots, n\} \\ &= \{2a_1 - 2r + (i + j)r \mid i, j = 1, 2, \dots, n\}, \end{aligned}$$

where $r \neq 0$ is the ratio. Thus $|S(A)| = |\{i + j \mid i, j = 1, 2, \dots, n\}| = 2n - 1$ (Proposition 1.3).

Reciprocally: Let $A = \{a_1, a_2, \dots, a_n\}$ with $a_1 < a_2 < \dots < a_n$ and assume that $|S(A)| = 2n - 1$.

The next $(2n - 1)$ sums:

$$\begin{aligned} S_2 &= a_1 + a_1, \quad S_3 = a_1 + a_2, \dots, \\ S_n &= a_1 + a_{n-1}, \quad S_{n+1} = a_1 + a_n, \\ S_{n+2} &= a_n + a_2, \quad S_{n+3} = a_n + a_3, \dots, \\ S_{2n-1} &= a_n + a_{n-1}, \quad S_{2n} = a_n + a_n \end{aligned}$$

are increasingly (distinct), so:

$$S(A) = \{S_2, S_3, \dots, S_{2n}\}.$$

We will show that for any other sum $a_i + a_j$ we have $a_i + a_j = S_{i+j}$, $i, j = 1, 2, \dots, n$.

• There are $2n - 1$ numbers:

$$\begin{aligned} a_1 + a_1 &< a_1 + a_2 < a_2 + a_2 < a_2 + a_3 < \dots < a_2 + a_n \\ &< a_n + a_3 < a_n + a_4 < \dots < a_n + a_n \end{aligned}$$

in an increasingly order, in $S(A)$, hence besides of above sums we have:

$$S_4 = a_2 + a_2, S_5 = a_2 + a_3, \dots, S_{n+2} = a_2 + a_n.$$

• There are $2n - 1$ distinct numbers:

$$\begin{aligned} a_1 + a_2 &< a_1 + a_2 < a_1 + a_3 < a_2 + a_3 < a_3 + a_3 < a_3 + a_4 < a_4 + a_4 \\ &< a_4 + a_5 < \dots < a_4 + a_n < a_n + a_5 < \dots < a_n + a_n, \end{aligned}$$

increasingly ordered, which are respectively:

$$S_2 < S_3 < S_4 < S_5 < S_6 < S_7 < S_8 < \dots < S_{2n},$$

and besides the sums already found we have:

$$S_8 = a_4 + a_4, S_9 = a_4 + a_5, \dots, S_{n+4} = a_4 + a_n.$$

Analogously we get $S_{i+j} = a_i + a_j$, $i, j = 1, 2, \dots, n$.

So $S_{2i} = a_i + a_i = a_{i-1} + a_{i+1}$, thus $a_i - a_{i-1} = a_{i+1} - a_i$, $i = 2, 3, \dots, n-1$ and $a_1, a_2, \dots, a_n$ are in arithmetic progression.

As an application of Theorem 2.1 we present Problem 4 which was given at the 60th Romanian Mathematical Olympiad 2009, grad X.

Problem. [4] Let $a_1 < a_2 < \dots < a_n$ and $b_1 < b_2 < \dots < b_m$, $m, n \in \mathbb{N}$, $m, n \geq 2$, be two sequences of real numbers and consider the set $\{a_i + b_j \mid 1 \leq i \leq n, 1 \leq j \leq m\}$. Prove that this set has $n + m - 1$ elements if and only if the sequences are in arithmetic progression with the same ratio.

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Theorem 2.2 *If $A = \{a_1, a_2, \dots, a_n\}$ and $B = \{b_1, b_2, \dots, b_m\}$ with $a_1 < a_2 < \dots < a_n$ and $b_1 < b_2 < \dots < b_m$, then:*

$$|A + B| = |A| + |B| - 1$$

if and only if the sequences $a_1, a_2, \dots, a_n$ and $b_1, b_2, \dots, b_m$ are arithmetic progression with the same ratio.

Proof. One of the implication is immediate, since for the common ratio d it follows

$$a_i = a_1 + (i - 1)d$$

and

$$b_j = b_1 + (j - 1)d,$$

hence

$$a_i + b_j = a_1 + b_1 + (i + j - 2)d,$$

in total we get $n + m - 1$ distinct values.

For the reverse implication, notice that

$$a_1 + b_1 < a_2 + b_1 < \dots < a_n + b_1 < a_n + b_2 < \dots < a_n + b_m,$$

hence the set of sums always contains at least $n + m - 1$ values.

We will use induction by $n + m$. The case $n = m = 2$ leads to

$$a_1 + b_1 < a_1 + b_2, \quad a_2 + b_1 < a_2 + b_2,$$

so $a_1 + b_2 = a_2 + b_1$, that is $a_2 - a_1 = b_2 - b_1$.

For $n + m > 4$, at least one of the values n, m (let it be m) is larger than 2. Then the sequence obtained by the elimination of b_m contains $m - 1$ elements, and we lose the sum $a_n + b_m$, maximum which will never be obtained, hence we have at most $n + m - 2$ possible values for the sums. By the first remark which we made, this is the minimum numbers of values, hence the answer is $n + (m - 1) - 1$. By induction hypothesis it follows that the sequence is in arithmetic progression with the same ratio. In a similar way, if we eliminate b_1 we lose the sum $a_1 + b_1$ (minimum which can not be obtained), with the same conclusion. But then the initial sequences are in arithmetic progression with the same ratio.

Remark 2.1 *If A, B are finite sets of positive numbers and*

$$A \cdot B = \{a \cdot b \mid a \in A, b \in B\},$$

then from Theorem 2.2 we obtain:

Theorem 2.3 *If $A = \{a_1, a_2, \dots, a_n\}$ and $B = \{b_1, b_2, \dots, b_m\}$ with $0 < a_1 < a_2 < \dots < a_n$ and $0 < b_1 < b_2 < \dots < b_m$, then:*

$$|A \cdot B| = |A| + |B| - 1$$

if and only if the sequences $a_1, a_2, \dots, a_n$ and $b_1, b_2, \dots, b_m$ are geometric progression with the same ratio.

3 The values of the map $|S|$

By Proposition 1.1 the values of the map $|S|$ are in the interval $\left[2|A| - 1, \frac{|A| \cdot (|A| + 1)}{2}\right]$ and we show that all these values can be taken.

Theorem 3.1 *For any positive integers n and for every positive integers $k \in \left[2n - 1, \frac{n(n+1)}{2}\right]$ there is a set A of real numbers (positive integers) for which the number of sums $|S(A)|$ is k .*

Proof. We will give a constructive proof. We start with the set $A = \{1, 2, \dots, n\}$ for which $|S(A)| = 2n - 1$ (minim), where we modify step by step one element until we obtain the set $B = \{1, 2, 2^2, \dots, 2^{n-1}\}$ for which $|S(B)| = \frac{n(n+1)}{2}$ (maxim). We define successively the sets A_k with

$$|S(A_k)| = k, \quad k = 2n - 1, 2n, \dots, \frac{n(n+1)}{2}.$$

Let

$$\begin{aligned}
A_{2n-1} &= \{1, 2, \dots, n-1, n\}, \\
S(A_{2n-1}) &= \{2, 3, \dots, 2n\} \\
A_{2n} &= \{1, 2, \dots, n-1, n+1\}, \\
S(A_{2n}) &= \{2, 3, \dots, 2n, 2n+2\} \\
A_{2n+1} &= \{1, 2, \dots, n-1, n+2\}, \\
S(A_{2n+1}) &= \{2, 3, \dots, 2n, 2n+1, 2n+4\} \\
A_{2n+2} &= \{1, 2, \dots, n-1, n+3\}, \\
S(A_{2n+2}) &= \{2, 3, \dots, 2n, 2n+1, 2n+2, 2n+6\} \\
&\dots\dots\dots \\
A_{3n-3} &= \{1, 2, \dots, n-1, 2(n-1)\}, \\
S(A_{3n-3}) &= \{2, 3, \dots, 3n-3, 4n-4\}.
\end{aligned}$$

If we change in A_{3n-3} the last term $2(n-1)$ with 2^{n-1} we continue by:

$$\begin{aligned}
A_{3n-3} &= \{1, 2, \dots, n-1, 2^{n-1}\}, \\
S(A_{3n-3}) &= [2, 2n-2] \cup [2^{n-1}+1, 2^{n-1}+n-1] \cup \{2^n\}.
\end{aligned}$$

Now we change $n-1$ in $n, n+1, \dots$

$$\begin{aligned}
A_{3n-2} &= \{1, 2, \dots, n-2, n, 2^{n-1}\} \\
A_{3n-1} &= \{1, 2, \dots, n-2, n+1, 2^{n-1}\} \\
A_{3n} &= \{1, 2, \dots, n-2, n+2, 2^{n-1}\} \\
&\dots\dots\dots \\
A_{4n-6} &= \{1, 2, \dots, n-2, 2(n-2), 2^{n-1}\}
\end{aligned}$$

in which we can change $2(n-2)$ by 2^{n-2} and we continue by the set

$$A_{4n-6} = \{1, 2, \dots, n-2, 2^{n-2}, 2^{n-1}\}.$$

We remark that

$$\begin{aligned} 2n - 1 &= n + (n - 1), \\ 3n - 3 &= n + (n - 1) + (n - 2), \\ 4n - 6 &= n + (n - 1) + (n - 2) + (n - 3). \end{aligned}$$

If we continue in same manner we get:

$$\begin{aligned} n + (n - 1) + (n - 2) + (n - 3) + (n - 4) &= 5n - 10 \text{ and} \\ A_{5n-10} &= \{1, 2, \dots, n - 3, 2^{n-3}, 2^{n-2}, 2^{n-1}\} \\ n + (n - 1) + (n - 2) + (n - 3) + (n - 4) + (n - 5) &= 6n - 15 \text{ and} \\ A_{6n-15} &= \{1, 2, \dots, n - 4, 2^{n-4}, 2^{n-3}, 2^{n-2}, 2^{n-1}\} \\ &\dots\dots\dots \\ \frac{n(n+1)}{2} &= n + (n - 1) + (n - 2) + \dots + 2 + 1 \text{ and} \\ A_{\frac{n(n+1)}{2}} &= \{1, 2, 2^2, \dots, 2^{n-1}\}. \end{aligned}$$

□

References

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