

Some remarks on risk functions described by utilities. Alternative proofs.

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ABSTRACT. In this paper we prove some properties of a generalized risk measure, defined by means of a convex, normalized, utility function. We also give an alternative proof for the subdifferential formula of this risk measure, by means of the infimal value function.

KEY WORDS: convex risk functions, utility function, optimized certainty equivalent

1 Introduction and notations

Nowadays a major challenge for both economics and financial mathematics is to generate risk measures by means of an utility model. A first approach was given by the seminal paper of Neumann and Morgenstern [7], where preference orders on random variables were introduced as a result of celebrated expected utility, i.e. $X \succeq Y$ if and only if $\mathbb{E}(u(X)) \geq \mathbb{E}(u(Y))$,

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where u is the utility function. Later Bühlmann et al. [5] introduced the principle of zero utility, described by means of M_u , the utility-mean, i.e. $\mathbb{E}(X - M_u(X)) = 0$. But the theory of utilities had known a rapidly grow due to the work of Ben-Tal and Teboulle [2], who introduced the Optimized Certainty Equivalent (i.e. the sure payment for which the decision maker remained indifferent to the lottery model), which successfully describes a wide class of risk functions by means of an concave utility function. This last approach was further developed and adapted by Boţ and Frăţean in [4], under the context of a convex utility model as this better suits the general framework of convex analysis and duality optimization. As follows we stay addicted to this last approach. In this paper we prove some properties of the generalized risk measures introduced in [4]. We present also an alternative proof for the subdifferential formula of this generalized risk measure.

In order to make the paper self-sufficient we start with a short presentation of the background we work under.

Let $\mathcal{X}$ be a Hausdoff locally convex vector space and $\mathcal{X}^*$ its topological dual space, endowed with the weak topology. $\langle x^*, x \rangle$ denotes the value of the linear continuous functional $x^* \in \mathcal{X}^*$ at $x \in \mathcal{X}$. The *indicator function* of a set $C \subseteq \mathcal{X}$ denoted by $\delta_C : \mathcal{X} \rightarrow \overline{\mathbb{R}}$, is

$$\delta_C(x) = \begin{cases} 0, & \text{if } x \in C, \\ +\infty, & \text{otherwise.} \end{cases}$$

We say that a function $f : \mathcal{X} \rightarrow \overline{\mathbb{R}}$ is *proper* if $\{x \in \mathcal{X} : f(x) < +\infty\} \neq \emptyset$ and $f(x) > -\infty$ for all $x \in \mathcal{X}$. For $f : \mathcal{X} \rightarrow \overline{\mathbb{R}}$ we define its (*Fenchel*) *conjugate* by $f^* : \mathcal{X}^* \rightarrow \overline{\mathbb{R}}$,

$$f^*(x^*) = \sup_{x \in \mathcal{X}} \{\langle x^*, x \rangle - f(x)\}.$$

Let $x \in \mathcal{X}$, the set $\partial f(x) = \{x^* \in \mathcal{X}^* : f(y) - f(x) \geq \langle x^*, y - x \rangle, \forall y \in \mathcal{X}\}$ denotes the (*convex*) *subdifferential* of f at x . The elements of the subdifferential are called *subgradients* of f . We say that the function f is subdifferentiable at x if $\partial f(x) \neq \emptyset$. If $f(x) \notin \mathbb{R}$ we consider by convention $\partial f(x) = \emptyset$. For a function $f : \mathcal{X} \rightarrow \overline{\mathbb{R}}$ the connection between the convex

subdifferential and the conjugate function is given by

$$(1.1) \quad x^* \in \partial f(x) \iff f(x) + f^*(x^*) = \langle x^*, x \rangle, \forall x \in \mathcal{X}.$$

Let, now $(\Omega, \mathfrak{F}, \mathbb{P})$, be an atomless probability space where Ω denotes the space of future states ω , $\mathfrak{F}$ is a σ -algebra on Ω and $\mathbb{P}$ is a *probability measure* on $(\Omega, \mathfrak{F})$. We treat as a random variable the function $X : \Omega \rightarrow \mathbb{R} \cup \{+\infty\}$ that belongs to the linear space

$$L^p(\Omega, \mathfrak{F}, \mathbb{P}) = \left\{ X : \Omega \rightarrow \mathbb{R} : X \text{ is measurable, } \int_{\Omega} |X(\omega)|^p d\mathbb{P}(\omega) < +\infty \right\},$$

$p \in [1, \infty]$. It is a known fact that the dual space of $L^p(\Omega, \mathfrak{F}, \mathbb{P}, \mathbb{R}) := L^p$, $p \in [1, \infty)$ is the space L^q , where $q \in (1, \infty]$ fulfills $q = p/(p-1)$, (with the convention $1/0 = \infty$). We consider that two functions define the same element of L^p when they are equal *almost everywhere* (a.e.). Also inequalities between random variables are to be seen in an almost everywhere way. For $X, Y \in L^p$ we write

$$X \geq Y \text{ if } X - Y \in L_+^p := \{X \in L^p : X \geq 0 \text{ a.e.}\}.$$

For a measurable random variable $X : \Omega \rightarrow \mathbb{R}$ the expected value with respect to $\mathbb{P}$ is defined by $\mathbb{E}(X) := \int_{\Omega} X(\omega) d\mathbb{P}(\omega)$. Whenever X takes the value $+\infty$ on a subset of positive measure we have $\mathbb{E}(X) = +\infty$. The *essential supremum* of X , which represents the smallest essential upper bound of the random variable, is $\text{essup } X := \inf\{a \in \mathbb{R} : \mathbb{P}(\omega : X(\omega) > a) = 0\}$, while its *essential infimum* is defined by $\text{esinf } X := -\text{essup}(-X)$.

Definition 1.1 We call risk function a proper function $\rho : L^p \rightarrow \overline{\mathbb{R}}$, $p \in [1, \infty]$. The risk function ρ is said to be

(i) convex, if:

$$\rho(\lambda X + (1 - \lambda)Y) \leq \lambda \rho(X) + (1 - \lambda)\rho(Y), \quad \forall \lambda \in [0, 1], X, Y \in L^p;$$

(ii) positively homogeneous, if: $\rho(0) = 0$ and

$$\rho(\lambda X) = \lambda \rho(X), \quad \forall \lambda > 0, X \in L^p;$$

- (iii) monotone, if: $X \geq Y \Rightarrow \rho(X) \leq \rho(Y)$, $\forall X, Y \in L^p$;
- (iv) expectation-bounded, if: $\rho(X) \geq -\mathbb{E}(X)$, $\forall X \in L^p$;
- (iv) cash-invariant, if: $\rho(X + a) = \rho(X) - a$, $\forall X \in L^p, a \in \mathbb{R}$.

For more on risk measures the reader is invited to see [6] (for convex risk measures), [1] (for coherent risk measures) and [10, 11] (for expectation bounded risk measures). The monotonic property (iii) in Definition 1.1 can be extended relatively to the first and second order stochastic dominance. For an economical interpretation of the monotonicity with respect to the stochastic order dominance we invite the reader to consult the book of Rachev et al. [8].

2 Properties of generalized convex risk measures

Definition 2.1 Let $u : \mathbb{R} \rightarrow \overline{\mathbb{R}}$ be a proper, convex, lower semicontinuous and nonincreasing function such that $u(0) = 0$ and $-1 \in \partial u(0)$. We define the generalized convex risk function $\rho_u : L^p \rightarrow \mathbb{R} \cup \{+\infty\}$, $p \in [1, \infty]$ by

$$(2.1) \quad \rho_u(X) = \inf_{\lambda \in \mathbb{R}} \{\lambda + \mathbb{E}(u(X + \lambda))\}.$$

The main properties of the above risk function are synthesized in the following results.

Proposition 2.1 The generalized risk function ρ_u described by relation (2.1) is a convex risk measure (i.e. it is convex, monotone and cash invariant risk function) bounded by the expectancy.

Proof.

To start with we prove the expectation boundedness property of ρ_u .

Expectation boundedness. Due Assumption, since $u(x) + x \geq 0$ for all $x \in \mathbb{R}$ one gets consequently $\mathbb{E}(u(X)) \geq -\mathbb{E}(X)$, for all $X \in L^p$, which yields the following chain of equalities

$$\begin{aligned} \rho_u(X) &= \inf_{\lambda \in \mathbb{R}} \{\lambda + \mathbb{E}(u(X + \lambda))\} \geq \inf_{\lambda \in \mathbb{R}} \{\lambda - \mathbb{E}(X + \lambda)\} = \\ &= \inf_{\lambda \in \mathbb{R}} \{\lambda - \mathbb{E}(X) - \lambda\} = -\mathbb{E}(X). \end{aligned}$$

Monotonicity. Let $X, Y \in L^p$ with $X(\omega) \leq Y(\omega)$ for a.e. $\omega \in \Omega$.

Since the utility function u is nonincreasing we get $u(X + \lambda) \geq u(Y + \lambda)$ for all $\lambda \in \mathbb{R}$. Taking into account the monotonicity of the expectancy we actually have $\lambda + \mathbb{E}(u(X + \lambda)) \geq \lambda + \mathbb{E}(u(Y + \lambda))$, $\forall \lambda \in \mathbb{R}$, which yields $\rho_u(X) \geq \rho_u(Y)$.

Cash invariance. In order to prove the translation invariance property of ρ_u we have the following chain of equalities.

$$\begin{aligned} \rho_u(X + a) &= \inf_{\eta \in \mathbb{R}} \{\eta + \mathbb{E}(u(X + a + \eta))\} \\ &= \inf_{\eta \in \mathbb{R}} \{\eta + a + \mathbb{E}(u(X + a + \eta))\} - a \\ &= \inf_{\substack{\lambda \in \mathbb{R} \\ \lambda = \eta + a}} \{\lambda + \mathbb{E}(u(X + \lambda))\} - a = \rho_u(X) - a, \forall a \in \mathbb{R}. \end{aligned}$$

Convexity. As for the convexity, let $X_1, X_2 \in L^p$ and $\alpha \in (0, 1)$. Since the utility function u is convex, the function $f : \mathbb{R} \times \mathbb{R} \rightarrow \mathbb{R}$ defined by $f(X, \lambda) = \lambda + u(X + \lambda)$ is convex over $\mathbb{R} \times \mathbb{R}$.

Let $X = \alpha X_1 + (1 - \alpha)X_2$. Then for $\lambda_1, \lambda_2 \in \mathbb{R}$ with $\lambda = \alpha \lambda_1 + (1 - \alpha)\lambda_2$ we have $\mathbb{E}(f(X, \lambda)) \leq \alpha \mathbb{E}(f(X_1, \lambda_1)) + (1 - \alpha)\mathbb{E}(f(X_2, \lambda_2))$. Passing to infimum in the above relation we get

$$\begin{aligned} \inf_{\lambda \in \mathbb{R}} \mathbb{E}(f(X, \lambda)) &\leq \inf_{\lambda_1, \lambda_2 \in \mathbb{R}} \{\alpha \mathbb{E}(f(X_1, \lambda_1)) + (1 - \alpha)\mathbb{E}(f(X_2, \lambda_2))\}, \\ \rho_u(X) &\leq \alpha \rho_u(X_1) + (1 - \alpha)\rho_u(X_2), \end{aligned}$$

which concludes the proof. ■

Remark 2.1 *Note that the above properties of ρ_u can be easily deduced from the corresponding properties of Optimized Certainly Equivalent [2, Theorem 2.1].*

Besides the basic properties stated in Proposition 2.1, the generalized convex risk function, ρ_u , may also exhibit stochastic dominance properties, i.e. (2.1) is monotonic with respect to the second order stochastic dominance. The conjugate and subdifferential formula for the generalized risk measure (2.1) were first obtained in [4]. We present here an alternative proof for the subdifferential formula by means of the so called *infimal value function*, meaningful for the duality approach.

Theorem 2.1 *The generalized risk function, ρ_u exhibits the following properties*

(i) *The conjugate function of ρ_u is the function $\rho_u^* : (L^p)^* \rightarrow \overline{\mathbb{R}}$, given by*

$$(2.2) \quad \rho_u^*(X^*) = \begin{cases} \mathbb{E}(u^*(X^*)), & \text{if } \mathbb{E}(X^*) = -1, \\ +\infty, & \text{otherwise.} \end{cases}$$

(ii) *Let $X \in L^p$ and $\bar{\lambda}(X) \in \mathbb{R}$ be the element where the infimum in the definition of $\rho_u(X)$ is attained. Then it holds*

$$(2.3) \quad \partial \rho_u(X) = \left\{ X^* \in (L^p)^* : \begin{array}{l} X^*(\omega) \in \partial u(X(\omega) + \bar{\lambda}(X)) \text{ a.e. } \omega \in \Omega, \\ \mathbb{E}(X^*) = -1 \end{array} \right\}.$$

Proof.

(i) By the definition of the conjugate function we get for all $X^* \in (L^p)^*$

$$\begin{aligned} \rho_u^*(X^*) &= \sup_{\substack{X \in L^p \\ \lambda \in \mathbb{R}}} \{ \langle X^*, X \rangle - \lambda - \mathbb{E}(u(X + \lambda)) \} \\ &= \sup_{\substack{R \in L^p \\ \lambda \in \mathbb{R}}} \{ \langle X^*, R - \lambda \rangle - \lambda - \mathbb{E}(u(R)) \} \\ &= \sup_{\lambda \in \mathbb{R}} \{ -\lambda(\mathbb{E}(X^*) + 1) \} + \sup_{R \in L^p} \{ \langle X^*, R \rangle - \mathbb{E}(u(R)) \}. \end{aligned}$$

Using the interchangeability property of minimization and integration (see [9, Theorem 14.60]) the second expression from above can be written as

$$\sup_{R \in L^p} \{ \langle X^*, R \rangle - \mathbb{E}(u(R)) \} = \mathbb{E} \left\{ \sup_{r \in \mathbb{R}} (rX^* - u(r)) \right\} = \mathbb{E}(u^*(X^*)).$$

On the other hand, since $\sup_{\lambda \in \mathbb{R}} \{ -\lambda(\mathbb{E}(X^*) + 1) \} = \delta_{\{0\}}(\mathbb{E}(X^*) + 1)$, one obtains the desired conclusion.

(ii) Applying [12, Theorem 2.6.2] the subdifferential formula for the infimal

value function ρ_u is

$$\partial\rho_u(X) = \bigcap_{\varepsilon>0} \bigcup_{\eta\in\mathbb{R}} \{X^* \in (L^P)^* : (X^*, 0) \in \partial_\varepsilon\phi(X, \eta)\},$$

where $\phi(X, \eta) = \eta + \mathbb{E}(v(X + \eta))$. Consequently, using (1.1) we obtain
(2.4)

$$\partial_\varepsilon\phi(X, \eta) = \{(X^*, \eta^*) : \phi(X, \eta) + \phi^*(X^*, \eta^*) \leq \langle X, X^* \rangle + \langle \eta, \eta^* \rangle + \varepsilon\}.$$

Now by similarly arguments as in the proof of the first assertion we obtain for the conjugate function of ϕ the following formula:

$$\begin{aligned} \phi^*(X^*, \eta^*) &= \sup_{\substack{X \in L^P \\ \eta \in \mathbb{R}}} \{\langle X^*, X \rangle + \langle \eta, \eta^* \rangle - \mathbb{E}(u(X + \eta))\} \\ &= \sup_{\eta \in \mathbb{R}} \{\eta(\eta^* - \mathbb{E}(X^*) - 1)\} + \sup_{r \in L^P} \{\langle X^*, r \rangle - \mathbb{E}(u(r))\} \\ &= \begin{cases} \mathbb{E}(u^*(X^*)), & \text{if } \mathbb{E}(X^*) + 1 = \eta^*; \\ +\infty, & \text{otherwise.} \end{cases} \end{aligned}$$

Thus (2.4) becomes,

$$\partial_\varepsilon\phi(X, \eta) = \left\{ (X^*, \eta^*) : \begin{array}{l} \mathbb{E}(u^*(X^*)) + \mathbb{E}(u(X + \eta)) + \eta \leq \\ \leq \langle X, X^* \rangle + \langle \eta, \eta^* \rangle + \varepsilon; \eta^* = \mathbb{E}(X^*) + 1 \end{array} \right\}.$$

The condition $(X^*, 0) \in \partial_\varepsilon\phi(X, \eta)$ can be written equivalently as

$$\mathbb{E}(u^*(X^*)) + \mathbb{E}(u(X + \eta)) + \eta \leq \langle X, X^* \rangle + \varepsilon.$$

Hence the monotonicity of the expected value function and the additional condition $\mathbb{E}(X^*) + 1 = 0$ lead to

$$\mathbb{E}(u^*(X^*)) + \mathbb{E}(u(X + \eta)) + \eta \leq \langle X, X^* \rangle + \varepsilon + \eta(\mathbb{E}(X^*) + 1),$$

which in view of relation (1.1) is equivalent with $X^* \in \partial_\varepsilon u(X + \eta)$. Consequently,

$$\partial\rho_u(X) = \bigcap_{\varepsilon>0} \bigcup_{\eta\in\mathbb{R}} \{X^* \in (L^P)^* : X^* \in \partial_\varepsilon u(X + \eta), \mathbb{E}(X^*) = -1\}.$$

It remains to prove that the above relation is nothing else than the set in the right-hand side of (2.3). Let then Z^* be an element in

$$\bigcap_{\varepsilon > 0} \bigcup_{\eta \in \mathbb{R}} \left\{ X^* \in (L^p)^* : \begin{array}{l} \mathbb{E}(u^*(X^*)) + \eta + u(X + \eta) \leq \langle X^*, X \rangle + \varepsilon, \\ \mathbb{E}(X^*) = -1 \end{array} \right\}$$

Then for all $k \geq 1$, when taking $\varepsilon_k := \frac{1}{k}$, there exists $\eta_k \in \mathbb{R}$ such that

$$\mathbb{E}(u^*(Z^*)) + \eta_k + u(X + \eta_k) \leq \langle Z^*, X \rangle + 1/k \text{ and } \mathbb{E}(Z^*) = -1.$$

Since the infimum of $\rho_u(X)$ is attained at $\bar{\lambda}(X)$, it follows

$$\mathbb{E}(u^*(Z^*)) + \bar{\lambda}(X) + u(X + \bar{\lambda}(X)) \leq \langle Z^*, X \rangle + 1/k$$

for all $k \geq 1$. Taking the limit when $k \rightarrow \infty$, one gets

$$\mathbb{E}(u^*(Z^*) + u(X + \bar{\lambda}(X)) - \langle Z^*, X + \bar{\lambda}(X) \rangle) \leq 0.$$

On the other hand, by Young-Fenchel's inequality, it holds

$$u^*(Z^*(\omega)) + u(X(\omega) + \bar{\lambda}(X)) - Z^*(\omega)(X(\omega) + \bar{\lambda}(X)) \geq 0, \quad \forall \omega \in \Omega.$$

Thus $\mathbb{E}(u^*(Z^*) + u(X + \bar{\lambda}(X)) - \langle Z^*, X + \bar{\lambda}(X) \rangle) = 0$, which means that

$$u^*(Z^*(\omega)) + u(X(\omega) + \bar{\lambda}(X)) - Z^*(\omega)(X(\omega) + \bar{\lambda}(X)) = 0$$

and, therefore, $Z^*(\omega) \in \partial u(X(\omega) + \bar{\lambda}(X))$ a.e. $\omega \in \Omega$.

Since the opposite inclusion is trivial, the proof follows.

■

Remark 2.2 *An important advantage when working with utility functions comes from the fact that, for different choices of the utility function u one rediscovers several well-known convex risk measures.*

a) *Consider the utility function $u_1 : \mathbb{R} \rightarrow \mathbb{R}$, $u_1(t) = \exp(-t) - 1$. The*

convex risk measure we define via u_1 is $\rho_{u_1} : L^p \rightarrow \mathbb{R}$,

$$\rho_{u_1}(X) = \inf_{\lambda \in \mathbb{R}} \{\lambda + \mathbb{E}(\exp(-X - \lambda) - 1)\} = \ln(\mathbb{E}(\exp(-X))),$$

known in the literature as the entropic risk measure.

b) By taking as utility function $u_2 = \delta_{[0, +\infty)}$ one rediscovers under $\rho_{u_2} : L^p \rightarrow \mathbb{R} \cup \{+\infty\}$,

$$\rho_{u_2}(X) = \inf_{\substack{\lambda \in \mathbb{R} \\ X + \lambda \geq 0}} \lambda = -\operatorname{esinf} X,$$

the so-called worst-case risk measure.

Consequently, corresponding conjugate and subdifferential formulae can be obtained (see [4]).

One of the major challenges of portfolio optimization is minimizing the risk of the portfolio while maximizing the profit. In recent portfolio optimization different risk and deviation measures have been considered (see [3]) for which optimality conditions have been obtained. Similarly one can deduce optimality conditions for a wider class of risk measures by working with the generalized risk measure (2.1) as objective function.

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