

Rational Asymmetric Approximation and Normal Functions

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ABSTRACT. This paper studies in the context of weighted asymmetric norm the classical problem of rational approximation of continuous real valued functions on a compact real interval. After extending the concept of normal functions we find by examples the normality condition of a function depends on the weights that define the norm. However it is proved here that main properties related to this concept in the rational uniform approximation still remain to be true in the asymmetric case.

KEY WORDS: Rational Approximation, Normal Function, Asymmetric Norm, Sign Sensitive Functional.

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1 Introduction

In their book [9] (first published in Russian in 1970), Krein and Nudel'man defined the concept of *asymmetric norm* $\|\cdot\|$ as a positive functional on a real linear space V that satisfies the triangle inequality and vanishes at $f \in V$ if and only if f is the zero vector. However the homogeneous property

$$\forall f \in V \quad \forall \alpha \in \mathbb{R}^+, \|\alpha f\| = \alpha \|f\|$$

is affirmed only for $\alpha \geq 0$ (i.e., for some $f \in V$, the inequality $\| -f \| \neq \| f \|$ is accepted). See a recent study on Asymmetric Functional Analysis in Cobzas book [2].

If $\phi \neq D \subset V$ and $f \in V$, we define the *best approximation to f from D* by

$$E = \inf \{ \|r - f\| : r \in D \},$$

and the *elements of best approximation* are those $r^* \in C$ for which

$$(1.1) \quad E = \|r^* - f\|.$$

In this paper we are interested on the study of *normal* functions related to *algebraic rational asymmetric approximation* (for shorten *raa* in the future) in the space $V = C[a, b]$, of continuous real valued functions on the real compact interval $[a, b]$, endowed with the asymmetric norm

$$(1.2) \quad \forall f \in C[a, b], \|f\|_{v_1, v_2} = \max_{x \in [a, b]} (f^+ v_2 + f^- v_1)(x),$$

where f^+ and f^- denote the positive and negative parts of the function f , and $v_1, v_2 \in C[a, b]$ are strictly positive weights. Due to these assumptions on the weights we shall count with the fundamental inequality of non degeneration

$$(1.3) \quad \exists K = K(v_1, v_2) > 0 \quad \forall f \in C[a, b], \| -f \|_{v_1, v_2} \leq K \|f\|_{v_1, v_2},$$

not available in general asymmetric normed spaces.

The functional (1.2) was called *sign sensitive functional* by Dolzhenko and Sebastyanov (see their large work on polynomial asymmetric

approximation and related results collected in [3]).

With respect to our study it must extend what we know in the context of classical rational uniform approximation (for shorten *rua* in the future), where the mathematical literature is large. See among others the representative book of Petrushev and Popov [16]. The case in which the data is a *discretization of the target function* (Kemp [7], [8]) is not considered here, although any discretization technique must be susceptible to a study under any asymmetric norm. The approximation by *generalized rational functions* will not be directly considered, however we shall use later several ideas on this subject developed by Cheney and Loeb in [1]).

Next section provides some definitions and facts as a necessary background for the understanding of what we are dealing with. It also contains in a natural way an extension of the concept of normal functions in dealing with *raa*, a few preliminary results, and the proof by mean of examples that the normality of a function depends on the weights. Section 3 contains the generalized results on normal functions that we have achieved to prove.

2 Rational asymmetric approximation

Denote by $\mathcal{P}_n$ the real linear space of algebraic polynomials with real coefficients up to degree $n \in \mathbb{N} \cup \{0\}$, and by $\mathcal{P}_n^*[a, b]$ its convex subset of polynomials q such that for every $x \in [a, b]$, $q(x) > 0$. Further, for given degrees n, m let be

$$R_{n,m}[a, b] := \{p/q : p \in \mathcal{P}_n \text{ and } q \in \mathcal{P}_m^*[a, b]\}.$$

We can suppose that $R_{n,m}[a, b] \subset C[a, b] = V$, and then take $R_{n,m}[a, b]$ as the approximating set D in V .

An important part of the present study of *raa* coalesces with previous results of Moursond with other authors where they introduce a generalized weight function (see [13], [14], [15], and [11], for instance). In particular Moursond proved the **existence** and **uniqueness** of the best algebraic *raa* with respect to the functional (1.2).

We remember that, if $r = p/q \in R_{n,m}[a, b]$ with $(p : q) = 1$ (i.e., irreducible between them except multiplicative constants), the *defect* of r in

$R_{n,m}[a, b]$ is the number

$$(2.1) \quad d_{n,m}(r) = \min \{n - \deg p, m - \deg q\} \text{ if } r \neq 0, m \text{ if } r = 0.$$

It is said that a function $g \in C[a, b]$ has k *alternation points* $\{x_i\}$ in $[a, b]$ with respect to (v_1, v_2) , if $x_i < x_{i+1}$, $i = 1, \dots, k-1$ and

$$(g^+v_2 - g^-v_1)(x_i) = (-1)^i \Theta \|g\|_{v_1, v_2}, \quad i = 1, \dots, k,$$

where $\Theta \in \{-1, 1\}$ is fixed.

The following result is in [15].

Theorem 2.1 [*Generalized Chebyshev-Akhiezer characterization*]
Let be $f \in C[a, b]$ and $r^* \in R_{n,m}[a, b]$. Then r^* is the best $\|\cdot\|_{v_1, v_2}$ approximation to f from $R_{n,m}[a, b]$, if and only if the function $r^* - f$ has at least $n + m + 2 - d_{n,m}(r^*)$ alternation points in $[a, b]$, with respect to (v_1, v_2) .

Note that under the assumed conditions on the weights any two asymmetric norms $\|\cdot\|_{u_1, u_2}$ and $\|\cdot\|_{v_1, v_2}$ are *equivalent* in the sense that

$$(2.2) \quad \forall u_1, u_2, v_1, v_2 \exists c_1, c_2 > 0 \forall f \in C[a, b], \quad c_1 \|f\|_{u_1, u_2} \leq \|f\|_{v_1, v_2} \leq c_2 \|f\|_{u_1, u_2}.$$

Thus, defining the open balls in any $(C[a, b], \|\cdot\|_{w_1, w_2})$ by

$$\forall f \in C[a, b] \quad \forall \varepsilon > 0, B_\varepsilon(f)_{w_1, w_2} = \left\{ g \in C[a, b] : \|g - f\|_{w_1, w_2} < \varepsilon \right\},$$

we find the remarkable fact all these asymmetric norms generate the same topology in $C[a, b]$.

Already knowing the existence, uniqueness, and a characterization of the elements of best *raa*, together with the equivalence (2.2), we conclude that most of the qualitative properties in the classical theory of *rua* (i.e., with $\|\cdot\|_{1,1}$) hold in dealing with any $\|\cdot\|_{v_1, v_2}$ asymmetric norm. But how calculate these elements of best *raa*? To count with a trustworthy algorithm for that would be useful. The natural candidate is the one-for-one exchange Remez algorithm whose convergence was already proved in the case of polynomial

asymmetric approximation [11] (see also [6]). However when we try to apply it to the *rua*, we discover that unexpected bad situations may appear during this process. Nevertheless extensions of the Remez algorithm for the *rua* are tried to employ often (see some related results in Ralston [17] and Werner [19]). In such cases the experience shows it is convenient not only appropriated started points to initiate the algorithm, but also other conditions that are satisfied when the target function is normal. In fact, normal functions play a relevant role in *rua* since they characterize the sets of continuity of the nonlinear operators of best approximation ([1], [12], and [20]). Up to our knowledge such study is missing in the context of the *raa* case. To fulfill this lack is the goal of this paper.

We must begin by extending in a natural way the definition of normal functions.

Definition 2.1 *We shall say that f is an (n, m, v_1, v_2) -normal function if the element r^* of best *raa* from $R_{n,m}[a, b]$ to f has defect zero.*

In other words that $r^* \notin R_{n-1,m-1}[a, b]$. Now we reveal the following examples. Let $f(x) = x$ be defined on $[-1, 1]$. The element of best rational $\|\cdot\|_{1,1}$ approximation to f from $R_{0,1}[-1, 1]$ is $r^*(x) \equiv 0$. The alternation points of $r^* - f$ are at $x_1 = -1$ and $x_2 = 1$ where the best *raa* is $E = 1$. The defect (2.1) is $d_{0,1}(r^*) = 1$ and consequently f is not $(0, 1, 1, 1)$ -normal function. On the other hand using the generalized characterization theorem of the element of best *raa* we check the best $\|\cdot\|_{\sqrt{1.01}/(\sqrt{1.01}-0.2), 1}$ *raa* to f from $R_{0,1}[-1, 1]$ is $t^*(x) \equiv (0.01) / (\sqrt{1.01} - x)$. The alternation points of $t^* - f$ are $x_1 = -1$, $x_2 = \sqrt{1.01} - 0.1$ and $x_3 = 1$. The best *raa* is $E = \sqrt{1.01}$. The defect is $d_{0,1}(t^*) = 0$ and consequently f is $(0, 1, \sqrt{1.01}/(\sqrt{1.01} - 0.2), 1)$ -normal.

Remark 2.1 *From these examples (and others in hands) we conclude that the normality condition of a fixed function with respect to the rational subset $R_{n,m}[a, b]$ **depends** on the weights v_1 and v_2 .*

3 The role of normal functions

The existence and uniqueness of the best *raa* allows us to define nonlinear operators

$$(3.1) \quad T_{v_1, v_2}^{n, m} : C[a, b] \rightarrow R_{n, m}[a, b]$$

that assign to each $f \in C[a, b]$, its element of best *raa* from $R_{n, m}[a, b]$ with respect to v_1, v_2 .

The main achievement of this section consists in characterizing the domain of continuity of the operators (3.1) in terms of normal functions. Due to the equivalence (2.2) it does not matter which $\|\cdot\|_{v_1, v_2}$ asymmetric norm we are dealing with for defining the topology of $C[a, b]$. However not only the definition of these operators strongly depends on the weights v_1, v_2 , but also the normality conditions of functions as we remarked at the end of the last section. Thus we need to answer the question which of the known theorems related to the continuity of the operator $T_{1, 1}^{n, m}$ still are true for the general case of *raa*. The somewhat surprising answer is “all the main ones”. It will be convenient to introduce the following notations:

$$(3.2) \quad N_{v_1, v_2}^{n, m} = N_{v_1, v_2}^{n, m}[a, b] = \{f \in C[a, b] : f \text{ is } (n, m, v_1, v_2) - \text{normal}\}$$

and

$$(3.3) \quad Q_{v_1, v_2}^{n, m} = Q_{v_1, v_2}^{n, m}[a, b] = C[a, b] \setminus N_{v_1, v_2}^{n, m}[a, b].$$

Proposition 3.1 [*Strong uniqueness*] *Let be $f \in N_{v_1, v_2}^{n, m}[a, b]$ and $r^* = T_{v_1, v_2}^{n, m}(f)$. Then*

$$\exists \gamma > 0 \ \forall r \in R_{n, m}[a, b], \ \|r - f\|_{v_1, v_2} \geq \|r^* - f\|_{v_1, v_2} + \gamma \|r - r^*\|_{1, 1}.$$

Proof. Our proof is rather long. But the reader can develop it by adapting to *raa* the proof of the corresponding result for uniform approximation of functions by generalized rational functions in [1]. Thus it will be omitted here. ■

We will integrate the early results for *rua* of Maehly and Witzgall [12], Cheney and Loeb [1], and Werner [20], into a general theorem in the context of *raa*. The proof follows the arguments in the original papers or in their later improvements, but also we should need some careful constructions. To this aim the equivalence method of asymmetric approximation sketched below (see details in Guerra-Jiménez [4] and Jiménez-Martínez [5]) provides a geometric help.

For any real $\lambda \geq 0$ set

$$M_\lambda = \{r \in R_{n,m}[a, b] : f - \lambda/v_1 \leq r \leq f + \lambda/v_2\},$$

and $\lambda^* = \inf \{\lambda \geq 0 : M_\lambda \neq \emptyset\}$. It can be proved this infimum is indeed a minimum. The sets M_λ do not vary continuously with λ . They have infinitely many elements for any $\lambda > \lambda^*$, $M_\lambda = \emptyset$ if $\lambda < \lambda^*$, while M_{λ^*} is a singleton set whose element r^* is the only one of best $\|\cdot\|_{v_1, v_2}$ approximation to f from $R_{n,m}[a, b]$. Except the trivial case $f \in R_{n,m}[a, b]$, in which $r^* = f$, the value λ^* is strictly positive. Following the generalized Chebyshev-Akhiezer characterization of the best approximation in terms of alternation sets of points, we find that geometrically r^* oscillates around f in the band with nonempty interior in the topology of $C[a, b]$, limited by the vertical lines $x = a$, $x = b$, and the inferior and superior borders $y = f - \lambda^*/v_1$ and $y = f + \lambda^*/v_2$ respectively. The characterization of r^* is expressed by the existence of ordered points x_i (perhaps not unique) in $[a, b]$, $1 \leq i \leq \chi$, where $\chi \geq n + m + 2 - d_{n,m}(r^*)$, at which

$$(3.4) \quad r^*(x_i) = (f - \lambda^*/v_1)(x_i) \text{ and } r^*(x_{i+1}) = (f + \lambda^*/v_2)(x_{i+1}),$$

or

$$(3.5) \quad r^*(x_i) = (f + \lambda^*/v_2)(x_i) \text{ and } r^*(x_{i+1}) = (f - \lambda^*/v_1)(x_{i+1}),$$

where $i = 1, 3, \dots$, and $i \leq \chi - 1$.

It will be also convenient to introduce the sets of classes $X_i, i = 1, 2, \dots, \chi$ as follows. Fix any maximum sequence of alternation points (x_i) as in (3.4)

y (3.5). If we are in the first case (3.4), as we shall suppose in what follows, set

$$(3.6) \quad \begin{aligned} X_i &= \left\{ x \in [a, b] : r^*(x) = \left(f - \frac{\lambda^*}{v_1} \right)(x), x_{i-1} < x < x_{i+1} \right\}, i = 1, 3, 5, \dots, \\ X_i &= \left\{ x \in [a, b] : r^*(x) = \left(f + \frac{\lambda^*}{v_2} \right)(x), x_{i-1} < x < x_{i+1} \right\}, i = 2, 4, 6, \dots, \end{aligned}$$

where the inequalities $x_j < x$ or $x < x_j$ are not taken into consideration if $j \notin \{1, 2, \dots, \chi\}$. The sets X_i are compact and disjoint between them. Moreover for each index i , $x_i \in X_i$. These sets are defined similarly in case (3.5).

Now we can prove the following lemmata.

Lemma 3.1 *If $f \in Q_{v_1, v_2}^{n, m} \setminus R_{n-1, m-1}$ in $[a, b]$ and $\varepsilon > 0$ is given, there is a function $g \in C[a, b]$, $g \neq f$, such that $T_{v_1, v_2}^{n, m}(g) = T_{v_1, v_2}^{n, m}(f)$ and $\|g - f\|_{1,1} < \varepsilon$.*

Proof. Since $f \neq T_{v_1, v_2}^{n, m}(f)$, there exists $t_0 \in [a, b]$ such that $t_0 \notin \cup_{i=1}^{\chi} X_i$. Choose X_{i^*} to be the nearest one to t_0 among the different X_i (if there are two such sets take one of them arbitrarily). Let l be the Euclidean distance between t_0 and X_{i^*} and let $s \in X_{i^*}$ be such that $|t_0 - s| = l$ (if there are two such points take one of them arbitrarily). Set $t_1 = t_0 + l/2$ if s lies to the right side of t_0 or $t_1 = t_0 - l/2$ if it lies to the left. Then define a continuous function H of Urysohn type on $[a, b]$ to be such that $0 \leq H(t) \leq 1$, $H(t_1) = 1$ and $H(t) = 0$ for every $t \notin [t_1 - l/4, t_1 + l/4]$. Let η be define as

$$\min \{ \psi_i(t) : t \in [t_i - l/4, t_i + l/4] \cap [a, b], \text{ and } i = 1, 2 \},$$

where $\psi_1(t) = r^*(t) - f(t) + E/v_1(t)$ and $\psi_2(t) = f(t) + E/v_2(t) - r^*(t)$. Take $0 < \mu < \min(\varepsilon, \eta)$ and then set $g = f + \mu H$. ■

Lemma 3.2 *Let $f \in Q_{v_1, v_2}^{n, m} \setminus R_{n-1, m-1}$ in $[a, b]$ be such that $d_{n, m}(T_{v_1, v_2}^{n, m}(f)) = \rho \geq 1$, and $\varepsilon > 0$ be given. Then there exists $g \in C[a, b]$ such that $d_{n, m}(T_{v_1, v_2}^{n, m}(g)) < \rho$ and $\|g - f\|_{1,1} < \varepsilon$.*

Proof. Set $r^* = T_{v_1, v_2}^{n, m}(f)$. From the hypothesis, $r^* \in R_{n-1, m-1}$, and $r^* - f$ has at least $n + m + 2 - \rho$ alternation points. Choose any X_{i_0} , among the sets X_i defined by (3.6), such that i_0 is neither the first nor the last of the index. We shall suppose X_{i_0} is one on which $r^*(x) = f(x) + E/v_2(x)$ (The other case can be treated similarly to this one). Since there are more than one but only a finite number of compact sets X_i , there exists an open set A , such that $X_{i_0} \subset A$ and $X_j \cap \bar{A} = \emptyset$ for every $j \neq i_0$, where $\bar{A}$ denotes the closure of A . By Urysohn lemma, choose a continuous real-valued function H on $[a, b]$, such that $0 \leq H \leq 1$, $H|_{X_{i_0}} \equiv 1$, and $H|_{A^c} \equiv 0$. Taking $0 < \mu < \varepsilon$ sufficiently small, we can define a function $g = f + \mu H$, such that $\|g - f\|_{1,1} < \varepsilon$, $g - E/v_1 \leq r^* \leq g + E/v_2$ and $r^*(x) - g(x) = 0$ if and only if $x \in X_j$, for some $j \neq i_0$.

The new alternation sets X'_i are $X'_i = X_i$ if $i < i_0 - 1$, $X'_{i_0-1} = X_{i_0-1} \cup X_{i_0+1}$, $X'_i = X_{i+2}$ for $i > i_0 - 1$, and from the generalized Chebyshev-Akhiezer characterization theorem for the best *raa*

$$T_{v_1, v_2}^{n-1, m-1}(g) = T_{v_1, v_2}^{n, m}(f) \neq T_{v_1, v_2}^{n, m}(g)$$

and then $d_{n, m}(T_{v_1, v_2}^{n, m}(g)) < \rho$. ■

Theorem 3.1 Fix n, m, v_1, v_2 . Then

- (i) The operator $T_{v_1, v_2}^{n, m}$ is continuous at $f \in C[a, b]$ if and only if $f \in R_{n, m}[a, b] \cup N_{v_1, v_2}^{n, m}[a, b]$.
- (ii) The set $N_{v_1, v_2}^{n, m}[a, b]$ is open and dense in $C[a, b]$, and its complement is a perfect subset of $C[a, b]$.

Proof. Fix any $\varepsilon > 0$. Let $f \in R_{n, m}[a, b]$, $g \in B_\varepsilon(f)_{v_1, v_2}$, and K the constant given by (1.3). The continuity of $T_{v_1, v_2}^{n, m}$ at f is immediately deduced from

$$\|T_{v_1, v_2}^{n, m}(g) - T_{v_1, v_2}^{n, m}(f)\|_{v_1, v_2} \leq \|T_{v_1, v_2}^{n, m}(g) - g\|_{v_1, v_2} + \|g - f\|_{v_1, v_2} \leq (1 + K)\varepsilon.$$

If $f \notin R_{n, m}[a, b]$ but $f \in N_{v_1, v_2}^{n, m}[a, b]$, we can use (2.2) and the constant γ given by the strong uniqueness to obtain the similar bound

$$\|T_{v_1, v_2}^{n, m}(g) - T_{v_1, v_2}^{n, m}(f)\|_{v_1, v_2} \leq c_1 \|T_{v_1, v_2}^{n, m}(g) - T_{v_1, v_2}^{n, m}(f)\|_{1,1} \leq c_1 (1 + K)\varepsilon/\gamma.$$

This proves one of the implications in (i). We shall retard the other one to the end in order to count with other affirmation not proved yet.

Using the property of local continuity of the operators $T_{v_1, v_2}^{n, m}$ at the functions in $N_{v_1, v_2}^{n, m}[a, b]$ and lemmas 5 and 6 in [1] for algebraic polynomials, we immediately obtain that the set $N_{v_1, v_2}^{n, m}[a, b]$ can be expressed as a union of preimages of open balls in $R_{n, m}[a, b]$ by continuous functions. Then $N_{v_1, v_2}^{n, m}[a, b]$ is open in $C[a, b]$, and on the other hand it is implied its complement $Q_{v_1, v_2}^{n, m}[a, b]$ is closed in $C[a, b]$. Let f be any of its elements. To prove that $Q_{v_1, v_2}^{n, m}[a, b]$ is perfect we need verify that f is an accumulation element of the same set $Q_{v_1, v_2}^{n, m}[a, b]$. If $f = p/q \in R_{n-1, m-1}[a, b]$, take any real sequence (σ_k) such that $\sigma_k > 1$, $\sigma_k \downarrow 1$ and $f_k = (\sigma_k p/q)$ to approximate f uniformly. If f is not a rational function in $R_{n, m}[a, b]$, it would be sufficient to show that given any $\varepsilon > 0$, we can define a function $g \in C[a, b]$, $g \neq f$, such that $T_{v_1, v_2}^{n, m}(g) = T_{v_1, v_2}^{n, m}(f)$ and $\|g - f\|_{1,1} < \varepsilon$. This is just the content of lemma 3.1.

To prove the density of $N_{v_1, v_2}^{n, m}$ in $C[a, b]$ with respect to the uniform norm, we choose any $f \in Q_{v_1, v_2}^{n, m}$, $\varepsilon' > 0$, and find the existence of $g \in N_{v_1, v_2}^{n, m}$, such that $\|g - f\|_{1,1} < \varepsilon'$.

We may omit the case $f = r \in R_{n-1, m-1}$ because as it is observed in [10] if $d_{n, m}(r) = \rho > 0$, then the sequence of (n, m, v_1, v_2) -normal functions

$$r(x) \left[\frac{x - t_k}{x - t_k - 1} \right]^\rho,$$

where $(t_k) \subset \mathbb{R}$ is an increasing not bounded sequence of positive real numbers such that t_1 is large enough, converges uniformly to r on $[a, b]$.

If $f \in Q_{v_1, v_2}^{n, m} \setminus R_{n, m}$, then by at most $\rho = d_{n, m}(T_{v_1, v_2}^{n, m}(f))$ times the application of lemma 3.2, with $\varepsilon = \varepsilon'/\rho$, we get the announced function g .

To prove the discontinuity of $T_{v_1, v_2}^{n, m}$ at any $f \notin R_{n, m} \sqcup N_{v_1, v_2}^{n, m}$ in $[a, b]$, set $r^* = T_{v_1, v_2}^{n, m}(f)$. Traditionally the proof is divided in two cases. First, the one given by Cheney and Loeb [1] when the maximum number χ of alternations sets of $r^* - f$ is equal or less than $n + m + 1$. Further, the result of Werner [20], when this number is equal or greater than $n + m + 2$. We can find both of them joined in Theorem 8.3, Chapter 7 of [10].

All properties used in [10] for proving the first case in the context of rua , have been already extended to raa . Thus we will omit to write here the proof of the case when $\chi \leq n + m + 1$. To prove the second case fix any $\varepsilon > 0$. It will be sufficient to construct a function $f_\varepsilon \in C[a, b]$, that satisfies the following affirmation: $\|f - f_\varepsilon\|_{1,1} < \varepsilon$ and $\|r^* - r_\varepsilon\|_{v_1, v_2} \geq \kappa > 0$, where $r_\varepsilon = T_{v_1, v_2}^{n, m}(f_\varepsilon)$ and κ is independent of ε .

In dealing with rua the usual proof involves a translation of $[a, b]$ to $[0, 1]$ that simplifies the problem, but for the case of raa this implies a possible modification of v_1 and v_2 that could need an extra analysis. So let us proceed as follows. Denote $\|r^* - f\|_{v_1, v_2}$ by E . Suppose that $a \notin X_1$, the first alternant set from the left whose first element will be denoted by x_1 . Then

$$f(a) - E/v_1(a) < r^*(a) < f(a) + E/v_2(a).$$

Since all involved functions are continuous there exists z , $a < z < x_1$, such that

$$\forall x \in [a, z], \quad f(x) - E/v_1(x) < r^*(x) < f(x) + E/v_2(x).$$

Write $\mu = \min_{x \in [a, z]}(f + E/v_2 - r^*)(x)$. Define the rational function $R(x) = B/(A(x - a) + l)q^*(x)$, with $B = q^*(a)\mu l$, $l = b - a$, and $A > 0$ large enough to get simultaneously R is decreasing on $[a, b]$ and $R(z) \leq \varepsilon$. Set $f_\varepsilon(x) = f(x) + R(z)$ on $[a, z]$, $f_\varepsilon(x) = f(x) + R(x)$ on $[z, b]$, and $r_\varepsilon = r^* + R$. Now it is a straightforward task to check out directly our affirmation.

If $a \in X_1$ but this first alternant set has other elements, after fixing $\varepsilon > 0$ we can easily redefine f around a to get a function $f' \in C[a, b]$ such that $a \notin X_1$, $\|f - f'\|_{1,1} < \varepsilon$ and $T_{v_1, v_2}^{n, m}(f') = T_{v_1, v_2}^{n, m}(f)$. If $X_1 = \{a\}$, we can redefine f as doing in the already quoted text [10]. ■

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