

Remarks on E -Strongly Convex Functions

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ABSTRACT. Starting from E -convex functions introduced by Youness and strongly convex functions introduced by Polyak we define and study E -strongly convex functions and E -strongly midconvex functions. We give a characterization of E -strongly convex functions using divided differences. We establish a connection between E -strongly midconvex functions and $E - \mathcal{F}$ -midconvex functions. A characterization of E -strongly midconvex functions in normed space is also done.

KEY WORDS: E -strongly convex functions, E -strongly midconvex functions

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1 Preliminary Notions and Results

The concepts of E -convex sets and E -convex functions have been introduced by Youness in [10]. Strongly convex functions have been introduced by Polyak in [7] and play an important role in optimization theory.

In [6] Lupşa gives a characterization of E -convex functions using divided differences. Some other properties of E -convex functions are also studied

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in [4]. In [5] Ger and Nikodem establish a characterization of strongly convex functions using divided differences. Starting from strongly convex functions with modulus c and from E -convex functions we define and study E -strongly convex functions with modulus c . We will give a characterization of E -strongly convex functions using divided differences.

Firstly we recall the definitions of E -convex sets, E -convex functions, strongly convex functions and strongly midconvex functions. Finally we recall the definitions of divided differences of order n . Using divided differences of order n , Popoviciu has introduced the notion of convex function of order n . A very important study of the properties of this class of functions can be found [9] and in [8].

Definition 1.1 [10] Let $E : \mathbb{R}^n \rightarrow \mathbb{R}^n$ be a function. A set $A \subset \mathbb{R}^n$ is called E -convex if $tE(x) + (1-t)E(y) \in A$, for every pair of points $x, y \in A$ and every $t \in [0, 1]$.

Definition 1.2 [10] Let $A \subset \mathbb{R}^n$ be a nonempty E -convex set. A function $f : A \rightarrow \mathbb{R}$ is called E -convex on A if

$$f(tE(x) + (1-t)E(y)) \leq tf(E(x)) + (1-t)f(E(y)),$$

for every pair of points $x, y \in A$ and every $t \in [0, 1]$.

Definition 1.3 [7] Let $I \subset \mathbb{R}$ be an interval and $c > 0$ be a real number. A function $f : I \rightarrow \mathbb{R}$ is called strongly convex with modulus c if

$$f(tx + (1-t)y) \leq tf(x) + (1-t)f(y) - ct(1-t)(x-y)^2$$

for every pair of points $x, y \in I$ and every $t \in [0, 1]$.

Definition 1.4 [7] Let $I \subset \mathbb{R}$ be an interval and $c > 0$ be a real number. A function $f : I \rightarrow \mathbb{R}$ is called strongly midconvex with modulus c if

$$f\left(\frac{x+y}{2}\right) \leq \frac{f(x)+f(y)}{2} - \frac{c}{4}(x-y)^2$$

for every pair of points $x, y \in I$.

We recall now the definition of divided differences. Let $A \subset \mathbb{R}$ be a nonempty set and let $f : A \rightarrow \mathbb{R}$ be a function. Let $n \in \mathbb{N}$ and let $x_1, x_2, \dots, x_{n+1}$ be distinct elements of A . Divided difference of order n on the points $x_1, x_2, \dots, x_{n+1}$ is denoted by $[x_1, x_2, \dots, x_{n+1}; f]$.

If $n = 0$, by definition

$$[x_1; f] = f(x_1),$$

and if $n \geq 1$, then the divided difference of order n is defined by

$$[x_1, x_2, \dots, x_{n+1}; f] = \frac{[x_2, x_3, \dots, x_{n+1}; f] - [x_1, x_2, \dots, x_n; f]}{x_{n+1} - x_1}.$$

2 E -Strongly Convex Functions and their Characterization by Divided Differences

Starting from strongly convex functions with modulus c and from E -convex functions we define and study E -strongly convex functions with modulus c .

Definition 2.1 Let $E : \mathbb{R} \rightarrow \mathbb{R}$ be a function and $A \subset \mathbb{R}$ be a nonempty E -convex set. Let $c > 0$ be a real number. A function $f : A \rightarrow \mathbb{R}$ is called E -strongly convex with modulus c if

$$(2.1) \quad \begin{aligned} & f(tE(x) + (1-t)E(y)) \\ & \leq tf(E(x)) + (1-t)f(E(y)) - ct(1-t)[E(x) - E(y)]^2 \end{aligned}$$

for every pair of points $x, y \in A$ and every $t \in [0, 1]$.

Theorem 2.1 Let $E : \mathbb{R} \rightarrow \mathbb{R}$ be an injective function and $A \subset \mathbb{R}$ be a nonempty E -convex set. Let $c > 0$ be a real number. A function $f : A \rightarrow \mathbb{R}$ is E -strongly convex with modulus c if and only if

$$[E(x), y, E(z); f] \geq c,$$

for every pair of points $x, z \in A$ and every $y \in (E(x), E(z))$, where $(E(x), E(z))$ is the open segment having the end points $E(x), E(z)$.

Proof. Let x and z be two arbitrary elements of the set A and let $y \in (E(x), E(z))$. Hence there is $t \in (0, 1)$ such that $y = tE(x) + (1-t)E(z)$.

From the definition of divided differences we have

$$\begin{aligned} [E(x), y, E(z); f] &= \frac{1}{E(z) - E(x)} \left[\frac{f(E(z)) - f(y)}{E(z) - y} - \frac{f(y) - f(E(x))}{y - E(x)} \right] \\ (2.2) \quad &= \frac{tf(E(x)) + (1-t)f(E(z)) - f(y)}{t(1-t)[E(z) - E(x)]^2} \end{aligned}$$

Necessity. If the function f is E -strongly convex with modulus c , then from (2.1) and (2.2) we obtain that $[E(x), y, E(z); f] \geq c$.

Sufficiency. If we have $[E(x), y, E(z); f] \geq c$, then (2.2) implies

$$tf(E(x)) + (1-t)f(E(z)) - f(y) \geq ct(1-t)[E(z) - E(x)]^2.$$

Hence

$$\begin{aligned} f(tE(x) + (1-t)E(z)) &\leq tf(E(x)) + (1-t)f(E(z)) \\ &\quad - ct(1-t)[E(z) - E(x)]^2. \end{aligned}$$

Since t is arbitrarily chosen in $(0, 1)$, we get that the function f is E -strongly convex with modulus c . ■

3 $E - \mathcal{F}$ -Midconvex Functions

We define E -strongly midconvex functions with modulus c and E -midconvex functions with respect to a two-parameter family $\mathcal{F}$. We establish a connection between E -strongly midconvex functions and E -midconvex functions with respect to $\mathcal{F}$, $\mathcal{F}$ conveniently chosen, similar to that obtained in [1] for strongly midconvex functions.

Definition 3.1 Let $E : \mathbb{R} \rightarrow \mathbb{R}$ be a function and $A \subset \mathbb{R}$ be a nonempty E -convex set. Let $c > 0$ be a real number. A function $f : A \rightarrow \mathbb{R}$ is called E -strongly midconvex with modulus c if (2.1) is satisfied for $t = \frac{1}{2}$, that is

$$f\left(\frac{E(x) + E(y)}{2}\right) \leq \frac{f(E(x)) + f(E(y))}{2} - \frac{c}{4} [E(x) - E(y)]^2$$

for every pair of points $x, y \in A$.

Let $\mathcal{F}$ be a family of continuous real functions defined on an a set $A \subset \mathbb{R}$. $\mathcal{F}$ is said to be a two-parameter family for A if for any two points $(x_1, y_1), (x_2, y_2) \in A \times \mathbb{R}$ with $x_1 \neq x_2$ there exists exactly one $\varphi \in \mathcal{F}$ such that

$$\varphi(x_i) = y_i, i = 1, 2.$$

The unique function $\varphi \in \mathcal{F}$ will be denoted by $\varphi_{(x_1, y_1), (x_2, y_2)}$.

If the set A is an interval, Beckenbach has introduced in [2] the notion of convex functions with respect to $\mathcal{F}$ and in [3] the notion of midconvex functions with respect to $\mathcal{F}$. In [8] E. Popoviciu has introduced the notion of convex function with respect to $\mathcal{F}$ in case the set $A \subset \mathbb{R}$ is not an interval.

Definition 3.2 [3] A function $f : A \rightarrow \mathbb{R}$ is called *midconvex with respect to $\mathcal{F}$* (shortly, *$\mathcal{F}$ -midconvex*) if

$$f\left(\frac{x_1 + x_2}{2}\right) \leq \varphi_{(x_1, f(x_1)), (x_2, f(x_2))}\left(\frac{x_1 + x_2}{2}\right)$$

for every pair of points $x_1, x_2 \in A$, $x_1 < x_2$.

We define now *E*-midconvex functions with respect to $\mathcal{F}$.

Definition 3.3 Let $E : \mathbb{R} \rightarrow \mathbb{R}$ be a function and $A \subset \mathbb{R}$ be a nonempty *E*-convex set. A function $f : A \rightarrow \mathbb{R}$ is called *E*-midconvex with respect to $\mathcal{F}$ (shortly, *E - $\mathcal{F}$ -midconvex*) if

$$f\left(\frac{E(x_1) + E(x_2)}{2}\right) \leq \varphi_{(E(x_1), f(E(x_1))), (E(x_2), f(E(x_2)))}\left(\frac{E(x_1) + E(x_2)}{2}\right)$$

for every pair of points $x_1, x_2 \in A$, $x_1 < x_2$.

Let $E : \mathbb{R} \rightarrow \mathbb{R}$ be a one to one function, $A \subset \mathbb{R}$ be a nonempty *E*-convex set and let $c > 0$ be a real number. Let $f : A \rightarrow \mathbb{R}$. We consider the two-parameter family $\mathcal{F}_c := \{cx^2 + ax + b \mid a, b \in \mathbb{R}\} \subset \mathbb{R}^A$.

We remark that if E is one-to-one on A , then the family $\mathcal{F}_c$ is a two-parameter family for $E(A)$.

Theorem 3.1 *Let $E : \mathbb{R} \rightarrow \mathbb{R}$ be a one to one function, $A \subset \mathbb{R}$ be a nonempty E -convex set and let $c > 0$ be a real number. A function $f : A \rightarrow \mathbb{R}$ is E -strongly midconvex convex with modulus c if and only if it is $E - \mathcal{F}_c$ -midconvex.*

Proof. Let $x_1, x_2 \in A$, $x_1 \neq x_2$. Since E is one-to-one on A , there exists a unique function $\varphi = \varphi_{(E(x_1), f(E(x_1))), (E(x_2), f(E(x_2)))} \in \mathcal{F}_c$, $\varphi(x) = cx^2 + ax + b$ such that $\varphi(E(x_i)) = f(E(x_i))$, $i = 1, 2$. Therefore

$$\begin{aligned} \varphi\left(\frac{E(x_1) + E(x_2)}{2}\right) &= c\left(\frac{E(x_1) + E(x_2)}{2}\right)^2 + a\frac{E(x_1) + E(x_2)}{2} + b \\ &= c\frac{E(x_1)^2 + 2E(x_1)E(x_2) + E(x_2)^2}{4} \\ &\quad + a\frac{E(x_1) + E(x_2)}{2} + b \\ &= \frac{1}{2}\left[cE(x_1)^2 + aE(x_1) + b\right] \\ &\quad + \frac{1}{2}\left[cE(x_2)^2 + aE(x_2) + b\right] \\ &\quad - \frac{c}{4}\left[E(x_1)^2 - 2E(x_1)E(x_2) + E(x_2)^2\right] \\ &= \frac{\varphi(E(x_1)) + \varphi(E(x_2))}{2} - \frac{c}{4}[E(x_1) - E(x_2)]^2 \\ &= \frac{f(E(x_1)) + f(E(x_2))}{2} - \frac{c}{4}[E(x_1) - E(x_2)]^2. \end{aligned}$$

Consequently f is E -strongly midconvex convex with modulus c if and only if it is $E - \mathcal{F}_c$ -midconvex. ■

4 Characterization of E -strongly midconvex functions in normed space

Definition 4.1 [7] *Let X be a normed space, $A \subset X$ be a convex subset of X and let $c > 0$ be a real number. A function $f : A \rightarrow \mathbb{R}$ is called strongly*

convex with modulus c if

$$(4.1) \quad f(tx + (1-t)y) \leq tf(x) + (1-t)f(y) - ct(1-t)\|x - y\|^2$$

for every pair of points $x, y \in A$ and every $t \in [0, 1]$

Definition 4.2 [7] Let X be a normed space, $A \subset X$ be a convex subset of X and let $c > 0$ be a real number. A function $f : A \rightarrow \mathbb{R}$ is called strongly midconvex convex with modulus c if (4.1) is satisfied for $t = \frac{1}{2}$, that is

$$f\left(\frac{x+y}{2}\right) \leq \frac{f(x)+f(y)}{2} - \frac{c}{4}\|x-y\|^2$$

for every pair of points $x, y \in A$.

Definition 4.3 Let X be a normed space, $E : X \rightarrow X$ be a function and $A \subset X$ be a nonempty E -convex set. A function $f : A \rightarrow \mathbb{R}$ is called E -midconvex if

$$f\left(\frac{E(x)+E(y)}{2}\right) \leq \frac{f(E(x))+f(E(y))}{2}$$

for every pair of points $x, y \in A$.

Definition 4.4 Let $E : X \rightarrow X$ be a function and $A \subset X$ be a nonempty E -convex set. Let $c > 0$ be a real number. A function $f : A \rightarrow \mathbb{R}$ is called E -strongly midconvex with modulus c if

$$f\left(\frac{E(x)+E(y)}{2}\right) \leq \frac{f(E(x))+f(E(y))}{2} - \frac{c}{4}\|E(x)-E(y)\|^2$$

for every pair of points $x, y \in A$.

We give a characterization of E -strongly midconvex function similar to that of strongly midconvex functions established in [1].

Theorem 4.1 Let X be a normed space whose norm satisfies the parallelogram law. Let $E : X \rightarrow X$ be a function and $A \subset X$ be a nonempty

E-convex set. Let $c > 0$ be a real number. A function $f : A \rightarrow \mathbb{R}$ is *E-strongly midconvex with modulus c* if and only if the function $g : A \rightarrow \mathbb{R}$

$$g(x) = f(x) - c\|x\|^2,$$

is *E-midconvex*.

Proof. We assume first that f is *E-strongly midconvex with modulus c* . We apply the Jordan-von Neumann parallelogram law and we have

$$\begin{aligned} g\left(\frac{E(x) + E(y)}{2}\right) &= f\left(\frac{E(x) + E(y)}{2}\right) - c\left\|\frac{E(x) + E(y)}{2}\right\|^2 \\ &\leq \frac{f(E(x)) + f(E(y))}{2} - \frac{c}{4}\|E(x) - E(y)\|^2 \\ &\quad - \frac{c}{4}\|E(x) + E(y)\|^2 \\ &= \frac{f(E(x)) + f(E(y))}{2} - \frac{c}{4}(2\|E(x)\|^2 + 2\|E(y)\|^2) \\ &= \frac{g(E(x)) + g(E(y))}{2} \end{aligned}$$

which proves that g is *E-midconvex*.

Conversely, if g is *E-midconvex*, then

$$\begin{aligned} f\left(\frac{E(x) + E(y)}{2}\right) &= g\left(\frac{E(x) + E(y)}{2}\right) + c\left\|\frac{E(x) + E(y)}{2}\right\|^2 \\ &\leq \frac{g(E(x)) + g(E(y))}{2} + c\left\|\frac{E(x) + E(y)}{2}\right\|^2 \\ &= \frac{g(E(x)) + c\|E(x)\|^2}{2} + \frac{g(E(y)) + c\|E(y)\|^2}{2} \\ &\quad + \frac{c}{4}(\|E(x) + E(y)\|^2 - 2\|E(x)\|^2 - 2\|E(y)\|^2) \\ &= \frac{f(E(x)) + f(E(y))}{2} - \frac{c}{4}\|E(x) - E(y)\|^2 \end{aligned}$$

hence f is *E-strongly midconvex with modulus c* . ■

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