

A new solution to Basel problem

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ABSTRACT. In this short note, we present a new proof of Euler's formula

$$1 + \frac{1}{2^2} + \frac{1}{3^2} + \cdots = \frac{\pi^2}{6}.$$

KEY WORDS: Basel problem, Fubini's theorem

MSC 2010: 40A25

1 Main Result

The famous Basel Problem, which asks for the precise sum of the series $\sum_{n=1}^{\infty} \frac{1}{n^2}$, was first posed by Pietro Mengoli in 1644 and remained open until 1735 when Leonhard Euler solved it. In the meantime, dozens and dozens of new proofs were provided (see the references).

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In our proof, we follow the techniques of [2], [6], [8], and [10], use the Fubini theorem for integrals and the series

$$\sum_{n=0}^{\infty} x^n = \frac{1}{1-x}, \quad |x| < 1.$$

We start from the double integral

$$I := \int_0^1 \int_0^{\infty} \frac{1}{1+y^2-x^2y^2} dy dx.$$

It follows that

$$(1.1) \quad I = \int_0^1 \int_0^{\infty} \frac{1}{1 + \left(\sqrt{1-x^2}\right)^2 y^2} dy dx = \int_0^1 \frac{\pi}{2\sqrt{1-x^2}} dx = \frac{\pi}{4}.$$

By changing the order of integration, we have:

$$I = \int_0^{\infty} \int_0^1 \frac{1}{1+y^2-x^2y^2} dx dy = \int_0^{\infty} \frac{\log\left(y + \sqrt{1+y^2}\right)}{y\sqrt{1+y^2}} dy$$

With the substitution

$$y = \frac{e^t - e^{-t}}{2},$$

we find that:

$$(1.2) \quad \begin{aligned} I &= \int_0^{\infty} \frac{\log\left(y + \sqrt{1+y^2}\right)}{y\sqrt{1+y^2}} dy \\ &= 2 \int_0^{\infty} \frac{te^{-t}}{1-e^{-2t}} dt = 2 \int_0^{\infty} t \sum_{n=0}^{\infty} e^{(-2n-1)t} dt \\ &= 2 \sum_{n=0}^{\infty} \int_0^{\infty} t e^{(-2n-1)t} dt = 2 \sum_{n=0}^{\infty} \frac{1}{(2n+1)^2} \\ &= 2 \left(\sum_{n=1}^{\infty} \frac{1}{n^2} - \sum_{n=1}^{\infty} \frac{1}{(2n)^2} \right) = \frac{3}{2} \sum_{n=1}^{\infty} \frac{1}{n^2} \end{aligned}$$

Eqs. (1.1) and (1.2) conclude the proof of Euler's formula.

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