

The dual gap function for an equilibrium problem

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ABSTRACT. Under appropriate assumptions we give a further class of gap functions for the general equilibrium problem introduced in the literature using a perturbation function, the so-called dual gap function. By particularizing this dual gap function we obtain several gap functions introduced in the literature for the Minty type variational inequality and equilibrium problem.

KEY WORDS: gap functions, dual gap functions, equilibrium problem, Fenchel duality, perturbation theory

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1 Introduction

Equilibrium problems provides an unified framework for the study of different problems in optimization, saddle and fixed point theory, variational inequalities, etc. One can notice that the so-called Minty variational inequality follows from the classical dual scheme for the equilibrium problem. The gap function was introduced by Hearn [14] and was interpreted as the difference between the cost function and the maximum of the Wolf dual of a convex optimization problem. Then, many authors point out a meaningful

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interpretation to the gap function by showing the connection between the gap function and duality of variational inequalities, respectively equilibrium problems, which corresponds to a pair of primal-dual Fenchel optimization problems [1, 2, 19, 17, 23]. Auslander introduced a gap function based on Fenchel duality (see [3]), while Giannessi, inspired by the Lagrange duality, introduced another gap function (see [13]).

In [1] the authors constructed gap function for equilibrium problem consisting in finding $x \in K$ such that

$$(EP) \quad F(x, y) \geq 0, \forall y \in K,$$

where X is a real topological vector space and $K \subseteq X$ is a nonempty closed and convex set, $F : X \times X \rightarrow \overline{\mathbb{R}}$ a bifunction satisfying $F(x, x) = 0, \forall x \in K$. The gap function is constructed by using Fenchel duality for the optimization problem attached to (EP) . In [1] the author also constructed under appropriate conditions gap function for the dual equilibrium problem consisting in finding $x \in K$ such that

$$(DEP) \quad F(y, x) \leq 0, \forall y \in K.$$

In [1] are given relations between the two gap functions and conditions such that the gap function constructed for (DEP) to be gap function for (EP) .

The aim of this article is to construct a dual gap function for the general equilibrium problem considered in [11] in formulation of which a so-called perturbation function is used.

2 Preliminaries

2.1 Notions and Results

Let us recall some notions and results useful in this paper. The notations are standard and follow [12, 22, 6, 7, 15, 21].

Consider X a real separated locally convex space and X^* its topological dual space. We denote by $w(X, X^*)$ ($w(X^*, X)$) the weak topology on X induced by X^* (the weak* topology on X^* induced by X). For a non-empty set $U \subseteq X$, we denote by $\text{cone}(U)$, $\text{aff}(U)$, $\text{lin}(U)$, $\text{int}(U)$, $\text{cl}(U)$, its *conical*

hull, *affine hull*, *linear hull*, *interior*, and *closure*, respectively. If $X = \mathbb{R}^n$ ($n \in \mathbb{R}$) is endowed with the Euclidean topology, the set $\text{ri}(U)$ denotes the classical *relative interior* of U , that is the interior of U relative to $\text{aff}(U)$. If $U \subseteq X$ a nonempty convex set we denote by $\text{sqri}(U) := \{x \in U : \text{cone}(U - x) \text{ is a closed linear subspace of } X\}$ the *strong quasi-relative interior* of U .

Let us consider $V \subseteq Y$ (Y being a real separated locally convex space) another nonempty set. The *projection operator* $\text{pr}_U : U \times V \rightarrow U$ is defined as $\text{pr}_U(u, v) = u$ for all $(u, v) \in U \times V$, while the *indicator function* of U , $\delta_U : X \rightarrow \overline{\mathbb{R}}$, is defined by $\delta_U(x) = 0$ if $x \in U$ and $+\infty$ otherwise (here $\overline{\mathbb{R}} = \mathbb{R} \cup \{\pm\infty\}$ is the extended real line).

For a function $f : X \rightarrow \overline{\mathbb{R}}$ we denote by $\text{dom } f = \{x \in X : f(x) < +\infty\}$ its *domain* and by $\text{epi } f = \{(x, r) \in X \times \mathbb{R} : f(x) \leq r\}$ its *epigraph*. We call f *proper* if $\text{dom } f \neq \emptyset$ and $f(x) > -\infty$ for all $x \in X$.

The *Fenchel-Moreau conjugate* of $f : X \rightarrow \overline{\mathbb{R}}$ is the function $f^* : X^* \rightarrow \overline{\mathbb{R}}$ defined by $f^*(x^*) = \sup_{x \in X} \{\langle x^*, x \rangle - f(x)\}$ for all $x^* \in X^*$ and the *biconjugate* function $f^{**} : X \rightarrow \overline{\mathbb{R}}$ is defined as $f^{**}(x) = \sup_{x^* \in X^*} \{\langle x^*, x \rangle - f^*(x^*)\}$ for all $x \in X$. Let us recall the *Young-Fenchel inequality*: $f^*(x^*) + f(x) \geq \langle x^*, x \rangle$ for all $x \in X$ and $x^* \in X^*$. Further, $f^{**} \leq f$ and according to the celebrated *Fenchel-Moreau Theorem*, if f is proper, then f is convex and lower semicontinuous if and only if $f^{**} = f$ (see [22, 6, 7, 12]).

For $x \in X$ such that $f(x) \in \mathbb{R}$ we define the *subdifferential* of f at x , by

$$\partial f(x) = \{x^* \in X^* : f(y) - f(x) \geq \langle x^*, y - x \rangle \ \forall y \in X\}.$$

If $f(x) \in \{\pm\infty\}$ we take by convention $\partial f(x) = \emptyset$. The *normal cone* of U at $x \in X$ is $N_U(x) = \{x^* \in X^* : \langle x^*, y - x \rangle \leq 0 \ \forall y \in U\}$, if $x \in U$ and $N_U(x) = \emptyset$ otherwise.

2.2 Conjugate dual problems by means of a perturbation function

In this section we recall some basic elements of duality theory needed later in the formulation of our problems.

In the literature were introduced (see [6, 7, 12, 22]) the conjugate dual problems by means of a general perturbation approach for different classes of primal problems. Considering a so-called perturbation function $\Phi : X \times Y \rightarrow$

$\overline{\mathbb{R}}$, where X and Y are supposed to be separated locally convex spaces, one can attach to the optimization problem

$$(PG) \quad \inf_{x \in X} \Phi(x, 0)$$

the following dual problem

$$(DG) \quad \sup_{y^* \in Y^*} \{-\Phi^*(0, y^*)\},$$

where $\Phi^* : X^* \times Y^* \rightarrow \overline{\mathbb{R}}$ is the conjugate function of Φ , while X^* and Y^* are the topological dual spaces of X and Y , respectively.

In order to ensure strong duality between the primal problem (PG) and its dual one (DG) , we need a regularity condition to be fulfilled. Some of the most important regularity conditions which ensure strong duality between the primal-dual problem pair are (see [6, 7, 22]):

$$(RC_1^\Phi) \quad \left| \begin{array}{l} \exists x' \in X \text{ such that } (x', 0) \in \text{dom } \Phi \text{ and } \Phi(x', \cdot) \text{ is} \\ \text{continuous at } 0; \end{array} \right.$$

$$(RC_2^\Phi) \quad \left| \begin{array}{l} X \text{ and } Y \text{ are Fréchet spaces, } \Phi \text{ is lower semicontinuous and} \\ 0 \in \text{int}(\text{pr}_Y(\text{dom } \Phi)); \end{array} \right.$$

$$(RC_3^\Phi) \quad \left| \begin{array}{l} X \text{ and } Y \text{ are Fréchet spaces, } \Phi \text{ is lower semicontinuous and} \\ 0 \in \text{sqli}(\text{pr}_Y(\text{dom } \Phi)); \end{array} \right.$$

and

$$(RC_4^\Phi) \quad \left| \begin{array}{l} \Phi \text{ is lower semicontinuous and } \text{Pr}_{X^* \times \mathbb{R}}(\text{epi } \Phi^*) \text{ is closed in} \\ (X^*, \omega(X^*, X)) \times \mathbb{R}. \end{array} \right.$$

In the setting of finite dimensional spaces one can consider also the following regularity condition

$$(RC_5^\Phi) \quad \left| \begin{array}{l} \dim(\text{lin}(\text{pr}_Y(\text{dom } \Phi))) < +\infty \text{ and } 0 \in \text{ri}(\text{pr}_Y(\text{dom } \Phi)). \end{array} \right.$$

Remark 2.1 *The conditions (RC_i^Φ) , $i \in \{1, 2, 3, 5\}$ are the so-called interiority-type regularity conditions, (RC_3^Φ) being an Attouch-Brézis type-one. (RC_4^Φ) is a closedness-type regularity condition, first introduced in [8, 9].*

Remark 2.2 *In case $\Phi : X \times Y \rightarrow \overline{\mathbb{R}}$ is a proper convex function such that $0 \in \text{pr}_Y(\text{dom } \Phi)$ and if one of the regularity conditions (RC_i^Φ) , $i \in \{1, 2, 3, 4, 5\}$ is fulfilled then stable strong duality holds (see [6, 7, 10, 16]):*

$$(2.1) \quad \sup_{x \in X} \{\langle x^*, x \rangle - \Phi(x, 0)\} = \min_{y^* \in Y^*} \{\Phi^*(x^*, y^*)\} \quad \forall x^* \in X^*.$$

Notice that in case $\Phi : X \times Y \rightarrow \overline{\mathbb{R}}$ is a proper, convex and lower semicontinuous function such that $0 \in \text{pr}_Y(\text{dom } \Phi)$, the relation (2.1) above is fulfilled if and only if (RC_4^Φ) holds. The relation

$$(\Phi(\cdot, 0))^*(x^*) \leq \inf_{y^* \in Y^*} \{\Phi^*(x^*, y^*)\}$$

for all $x^* \in X^*$ is fulfilled in most general framework.

3 Dual gap function for the general equilibrium problem

In [11] the authors introduced gap functions for a more general equilibrium problem in the sense of Stampacchia, by using a perturbation function Φ , which consists in finding a point $\bar{x} \in X$ such that

$$(PEP) \quad F(\bar{x}, x) + \Phi(x, 0) \geq \Phi(\bar{x}, 0), \forall x \in X,$$

where X, Y are real separated locally convex spaces, $\Phi : X \times Y \rightarrow \overline{\mathbb{R}}$ is a proper function fulfilling $0 \in \text{pr}_Y(\text{dom } \Phi)$ and $F : X \times X \rightarrow \overline{\mathbb{R}}$ is a bifunction satisfying $F(x, x) = 0, \forall x \in \text{dom } \Phi(\cdot, 0)$.

A function $\gamma : X \rightarrow \overline{\mathbb{R}}$ is said to be a gap function for the problem (PEP) if it satisfies the following properties:

- (i) $\gamma(x) \geq 0 \quad \forall x \in X$;
- (ii) $\gamma(x) = 0$ if and only if x solves the problem (PEP).

The function $\gamma^{PEP} : X \rightarrow \overline{\mathbb{R}}$ defined for all $\bar{x} \in \text{dom } \Phi(\cdot, 0)$ by

$$\gamma^{PEP}(\bar{x}) = \inf_{x^* \in X^*, y^* \in Y^*} \{F_x^*(\bar{x}, x^*) + \Phi^*(-x^*, y^*)\} + \Phi(\bar{x}, 0)$$

and $\gamma^{PEP}(\bar{x}) = +\infty$ for $\bar{x} \notin \text{dom } \Phi(\cdot, 0)$ was constructed by using Fenchel duality for the optimization problem attached to (PEP) , where $F_x^*(x, \cdot)$ denotes the conjugate of F with respect to the second variable, i.e. $F_x^*(\bar{x}, x^*) = (F(\bar{x}, \cdot))^*(x^*)$. It was proved that under convexity hypotheses and the fulfillment of a regularity condition, γ^{PEP} becomes a gap function for (PEP) .

In this section we introduce another class of gap functions for the problem (PEP) . In what follows we deal with the so-called dual equilibrium problems (Minty type) which is closely related to (PEP) and consists in finding $\bar{x} \in X$ such that

$$(DPEP) \quad F(x, \bar{x}) + \Phi(\bar{x}, 0) \leq \Phi(x, 0), \forall x \in X.$$

We denote by S^{PEP} and S^{DPEP} the solution set of the problems (PEP) and $(DPEP)$ respectively.

In order to formulate another gap function for (PEP) using the dual equilibrium problem $(DPEP)$ we recall some definitions and we give some results (see [4, 5, 17, 20]).

Definition 3.1 *The bifunction $F : X \times X \rightarrow \mathbb{R}$ is said to be*

(i) **monotone** *if, for each pair of points $x, y \in X$, we have*

$$F(x, y) + F(y, x) \leq 0;$$

(ii) **pseudomonotone** *if, for each pair of points $x, y \in X$, we have*

$$F(x, y) \geq 0 \text{ implies } F(y, x) \leq 0.$$

Definition 3.2 *The function $\varphi : X \rightarrow \mathbb{R}$ is said to be*

(i) **quasiconvex** *if, for each pair of points $x, y \in X$ and for all $\alpha \in [0, 1]$, we have*

$$\varphi(\alpha x + (1 - \alpha)y) \leq \max\left\{\varphi(x), \varphi(y)\right\};$$

- (ii) **explicitly quasiconvex** if it is quasiconvex and for each pair of points $x, y \in X$ such that $\varphi(x) \neq \varphi(y)$ and all $\alpha \in (0, 1)$, we have

$$\varphi(\alpha x + (1 - \alpha)y) < \max\left\{\varphi(x), \varphi(y)\right\};$$

- (iii) **(explicitly) quasiconcave** if $-\varphi$ is (explicitly) quasiconvex.

Definition 3.3 (i) The function $\varphi : X \rightarrow \mathbb{R}$ is said to be **u-hemicontinuous** if, for all $x, y \in X$ and $\alpha \in [0, 1]$, the function $\tau(\alpha) = \varphi(\alpha x + (1 - \alpha)y)$ is upper semicontinuous at 0.

- (ii) The function $\varphi : X \rightarrow \mathbb{R}$ is said to be **l-hemicontinuous** if, for all $x, y \in X$ and $\alpha \in [0, 1]$, the function $\tau(\alpha) = \varphi(\alpha x + (1 - \alpha)y)$ is lower semicontinuous at 0.

In [18] the authors showed that in appropriate hypothesis like monotonicity, convexity, hemicontinuity we can establish inclusions between the solution set of the Stampacchia type equilibrium problem and the solution set of the corresponding Minty type equilibrium problem. The proofs of Proposition 3.1 and 3.2 below use some well-known techniques in this framework.

Proposition 3.1 If F is a monotone bifunction, then $S^{PEP} \subseteq S^{DPEP}$.

Proof. Let $\bar{x} \in S^{PEP}$, that is $F(\bar{x}, x) + \Phi(x, 0) \geq \Phi(\bar{x}, 0), \forall x \in X$. Using the monotonicity of F we have that $0 \leq F(\bar{x}, x) + \Phi(x, 0) - \Phi(\bar{x}, 0) \leq -F(x, \bar{x}) + \Phi(x, 0) - \Phi(\bar{x}, 0), \forall x \in X$. Hence $\bar{x}$ is a solution of (DPEP) and $\bar{x} \in S^{DPEP}$.

Proposition 3.2 Let $F(x, \cdot)$ convex $\forall x \in X$, $F(\cdot, x)$ u-hemicontinuous $\forall x \in X$ and $\Phi(\cdot, 0)$ be proper, convex and l-hemicontinuous. Then $S^{DPEP} \subseteq S^{PEP}$.

Proof. Let $\bar{x} \in S^{DPEP}$. For a fixed $x \in X$ we put $x_t = tx + (1 - t)\bar{x}$ for each $t \in (0, 1)$, and by (DPEP) we obtain

$$(3.1) \quad F(x_t, \bar{x}) + \Phi(\bar{x}, 0) \leq \Phi(x_t, 0), \forall t \in (0, 1).$$

Let us suppose that for one $t \in (0, 1)$ it holds $F(x_t, x) + \Phi(x, 0) < \Phi(x_t, 0)$. We have

$$\begin{aligned}
\Phi(x_t, 0) &\leq F(x_t, x_t) + \Phi(x_t, 0) \\
&= F(x_t, tx + (1-t)\bar{x}) + \Phi(tx + (1-t)\bar{x}, 0) \\
&\quad (\text{by convexity of } F(x, \cdot) \text{ and } \Phi(\cdot, 0)) \\
&\leq tF(x_t, x) + (1-t)F(x_t, \bar{x}) + t\Phi(x, 0) + (1-t)\Phi(\bar{x}, 0) \\
&= t(F(x_t, x) + \Phi(x, 0)) + (1-t)(F(x_t, \bar{x}) + \Phi(\bar{x}, 0)) \\
&\quad (\text{by supposition}) \\
&< t(\Phi(x_t, 0)) + (1-t)(F(x_t, \bar{x}) + \Phi(\bar{x}, 0)) \\
&\quad (\text{by (3.1)}) \\
&\leq t\Phi(x_t, 0) + (1-t)\Phi(x_t, 0) = \Phi(x_t, 0)
\end{aligned}$$

and we obtain a contradiction. Then, we have that $F(x_t, x) + \Phi(x, 0) \geq \Phi(x_t, 0), \forall t \in (0, 1)$. By using l-hemicontinuity of $\Phi(\cdot, 0)$ and u-hemicontinuity of $F(\cdot, x)$ we have

$$\begin{aligned}
\Phi(\bar{x}, 0) &\leq \liminf_{t \searrow 0} \Phi(x_t, 0) \leq \liminf_{t \searrow 0} [F(x_t, x) + \Phi(x, 0)] \\
&\leq \limsup_{t \searrow 0} [F(x_t, x) + \Phi(x, 0)] \leq F(\bar{x}, x) + \Phi(x, 0)
\end{aligned}$$

hence $\bar{x}$ is a solution of (PEP) i.e. $\bar{x} \in S^{PEP}$.

Remark 3.1 *Let us notice that we can replace the convexity of the functions $F(x, \cdot)$ and $\Phi(\cdot, 0)$ in Proposition 3.2 with another convex-type condition such that the inclusion $S^{DPEP} \subseteq S^{PEP}$ is fulfilled. Under the hypotheses of Proposition 3.2 if we replace the convexity of the functions $F(x, \cdot)$ and $\Phi(\cdot, 0)$ with explicitly quasiconvexity of the function $F(x, \cdot) + \Phi(\cdot, 0)$, we can also prove the inclusion $S^{DPEP} \subseteq S^{PEP}$.*

In this case we have

$$(3.2) \quad F(x_t, x_t) + \Phi(x_t, 0) \leq \max\{F(x_t, x) + \Phi(x, 0), F(x_t, \bar{x}) + \Phi(\bar{x}, 0)\}$$

where $x_t = tx + (1-t)\bar{x}, \forall t \in (0, 1)$. Suppose now that there exists $t \in (0, 1)$ such that $F(x_t, x) + \Phi(x, 0) < F(x_t, \bar{x}) + \Phi(\bar{x}, 0)$. This yields that $F(x_t, x_t) + \Phi(x_t, 0) < F(x_t, \bar{x}) + \Phi(\bar{x}, 0)$ which is a contradiction since $\bar{x}$ is a

solution of $(DPEP)$. So, $F(x_t, x) + \Phi(x, 0) \geq F(x_t, \bar{x}) + \Phi(\bar{x}, 0), \forall t \in (0, 1)$, and using (3.2) we obtain that $F(x_t, x) + \Phi(x, 0) \geq \Phi(x_t, 0)$. By letting $t \searrow 0$, u -hemicontinuity of $F(\cdot, x)$ and l -hemicontinuity of $\Phi(\cdot, 0)$ yields $F(\bar{x}, x) + \Phi(x, 0) \geq \Phi(\bar{x}, 0)$ which means that $\bar{x}$ is a solution for (PEP) .

To the dual equilibrium problem $(DPEP)$ one can attach the following optimization problem:

$$(P^{DPEP}, \bar{x}) \quad \inf_{x \in X} \{-F(x, \bar{x}) + \Phi(x, 0)\} - \Phi(\bar{x}, 0)$$

where $\bar{x}$ is fixed. The Fenchel dual problem to $(P^{DPEP}, \bar{x})$ is (see [12, 22]):

$$(D^{DPEP}, \bar{x}) \quad \sup_{x^* \in X^*} \left\{ -\sup_{x \in X} [\langle x^*, x \rangle + F(x, \bar{x})] - (\Phi(\cdot, 0))^*(-x^*) \right\} - \Phi(\bar{x}, 0).$$

Notice that, since $(\Phi(\cdot, 0))^*(-x^*) \leq \inf_{y^* \in Y^*} \Phi^*(-x^*, y^*)$, we have $v(P^{DPEP}, \bar{x}) \geq v(D^{DPEP}, \bar{x}) \geq v(D'^{DPEP}, \bar{x})$, where

$$(D'^{DPEP}, \bar{x}) \quad \sup_{x^* \in X^*, y^* \in Y^*} \left\{ -\sup_{x \in X} [\langle x^*, x \rangle + F(x, \bar{x})] - \Phi^*(-x^*, y^*) \right\} - \Phi(\bar{x}, 0).$$

Following [1, 2], we introduce now the function γ^{DPEP} defined for all $\bar{x} \in \text{dom } \Phi(\cdot, 0)$ by

$$\begin{aligned} \gamma^{DPEP}(\bar{x}) &:= -v(D'^{DPEP}, \bar{x}) = \\ &= \inf_{x^* \in X^*, y^* \in Y^*} \left\{ \sup_{x \in X} [\langle x^*, x \rangle + F(x, \bar{x})] + \Phi^*(-x^*, y^*) \right\} + \Phi(\bar{x}, 0), \end{aligned}$$

and $\gamma^{DPEP}(\bar{x}) = +\infty$ for $\bar{x} \notin \text{dom } \Phi(\cdot, 0)$.

Let us recall [11, Theorem 4.1] which affirm that under convexity hypotheses and the fulfillment of a regularity condition $\gamma^{PEP}(\bar{x})$ becomes a gap function for (PEP) .

[11, Theorem 4.1] *Let X and Y be real separated locally convex spaces, $\Phi : X \times Y \rightarrow \overline{\mathbb{R}}$ a proper and convex function, $F : X \times X \rightarrow \overline{\mathbb{R}}$ a proper bifunction such that for all $\bar{x} \in \text{dom } \Phi(\cdot, 0)$, $F(\bar{x}, \bar{x}) = 0$,*

$\text{dom } \Phi(\cdot, 0) \cap \text{dom } F(\bar{x}, \cdot) \neq \emptyset$ and $F(\bar{x}, \cdot)$ is convex and continuous at a point in $\text{dom } \Phi(\cdot, 0) \cap \text{dom } F(\bar{x}, \cdot)$. Assume that one of the regularity conditions (RC_i^Φ) , $i \in \{1, 2, 3, 4, 5\}$ is fulfilled. Then γ^{PEP} is a gap function for the problem (PEP).

By using this theorem for (DPEP) we can affirm that if the function $y \mapsto -F(y, \bar{x})$ is convex and continuous at a point x in $\text{dom } \Phi(\cdot, 0) \cap \text{dom } F(\bar{x}, \cdot)$ and assume that one of the regularity conditions (RC_i^Φ) , $i \in \{1, 2, 3, 4, 5\}$ is fulfilled, then γ^{DPEP} is gap function for the problem (DPEP).

Corollary 3.1 *Let X and Y be real separated locally convex spaces, $\Phi : X \times Y \rightarrow \overline{\mathbb{R}}$ a proper and convex function, $F : X \times X \rightarrow \overline{\mathbb{R}}$ a proper bifunction such that for all $\bar{x} \in \text{dom } \Phi(\cdot, 0)$, $F(\bar{x}, \bar{x}) = 0$, $\text{dom } \Phi(\cdot, 0) \cap \text{dom } F(\bar{x}, \cdot)$ and $-F(\cdot, \bar{x})$ is convex and continuous at a point x in $\text{dom } \Phi(\cdot, 0) \cap \text{dom } F(\bar{x}, \cdot)$. Assume that one of the regularity conditions (RC_i^Φ) , $i \in \{1, 2, 3, 4, 5, 7\}$ is fulfilled. Then γ^{DPEP} is gap function for the problem (DPEP).*

We can compare now the function γ^{PEP} with the new one introduced, namely γ^{DPEP} .

Proposition 3.3 *Assume that F is monotone bifunction. Then it holds*

$$\gamma^{DPEP}(x) \leq \gamma^{PEP}(x), \forall x \in X.$$

Proof. By the monotonicity of F , we have

$$F(x, \bar{x}) + F(\bar{x}, x) \leq 0, \forall x, \bar{x} \in X,$$

or equivalently $F(x, \bar{x}) \leq -F(\bar{x}, x), \forall x, \bar{x} \in X$. Adding $\langle x^*, x \rangle$ and taking supremum in both sides over all $x \in X$ yields

$$\sup_{x \in X} \{ \langle x^*, x \rangle + F(x, \bar{x}) \} \leq F_x^*(\bar{x}, x^*).$$

Adding now $\Phi^*(-x^*, y^*)$, taking infimum on both sides over all $x^* \in X^*, y^* \in Y^*$ and then adding $\Phi(\bar{x}, 0)$ we have

$$\begin{aligned} & \inf_{x^* \in X^*, y^* \in Y^*} \left\{ \sup_{x \in X} [\langle x^*, x \rangle + F(x, \bar{x})] + (\Phi^*(-x^*, y^*)) \right\} + \Phi(\bar{x}, 0) \\ & \leq \inf_{x^* \in X^*, y^* \in Y^*} \{ F_x^*(\bar{x}, x^*) + (\Phi^*(-x^*, y^*)) + \Phi(\bar{x}, 0), \forall x \in X, \end{aligned}$$

and the proof is complete.

In what follows we give conditions for which the function γ^{DPEP} becomes gap function for the general equilibrium problem in sense of Stampacchia, (PEP) .

Theorem 3.1 *Let X and Y be real separated locally convex spaces, $\Phi : X \times Y \rightarrow \overline{\mathbb{R}}$ a proper and convex function, $\Phi(\cdot, 0)$ l -hemicontinuous, $F : X \times X \rightarrow \overline{\mathbb{R}}$ a proper and monotone bifunction such that for all $\bar{x} \in \text{dom } \Phi(\cdot, 0)$, $F(\bar{x}, \bar{x}) = 0$, $F(x, \cdot)$ convex $\forall x \in X$, $\text{dom } \Phi(\cdot, 0) \cap \text{dom } F(\bar{x}, \cdot)$, $F(\cdot, \bar{x})$ is convex and continuous at a point $x \in \text{dom } \Phi(\cdot, 0) \cap \text{dom } F(\bar{x}, \cdot)$ and $F(\cdot, x)$ is u -hemicontinuous $\forall x \in X$. Assume that one of the regularity conditions (RC_i^Φ) , $i \in \{1, 2, 3, 4, 5\}$ is fulfilled. Then γ^{DPEP} is gap function for (PEP) .*

Proof.

(i) By the weak duality it holds

$$\begin{aligned} \gamma^{DPEP}(\bar{x}) &= -v(D'^{DPEP}, \bar{x}) \geq -v(D^{DPEP}, \bar{x}) \geq \\ &\geq -v(P^{DPEP}, \bar{x}) \geq 0, \forall \bar{x} \in X. \end{aligned}$$

(ii) " $\Rightarrow$ " If $\gamma^{DPEP}(\bar{x}) = 0$, then $0 = v(D'^{DPEP}, \bar{x}) \leq v(D^{DPEP}, \bar{x}) \leq v(P^{DPEP}, \bar{x}) \leq 0$, i.e. $v(P^{DPEP}, \bar{x}) = 0$ and $\bar{x}$ is a solution for $(P^{DPEP}, \bar{x})$ or $\bar{x}$ solves $(DPEP)$. Applying the Proposition 3.2 we have that $\bar{x}$ solves (PEP) .

" $\Leftarrow$ " We consider $\bar{x}$ a solution of (PEP) . In this case we have that $\gamma^{PEP}(\bar{x}) = 0$ (cf. Theorem 4.1 in [11]). By Proposition 3.3 we have that $\gamma^{DPEP}(\bar{x}) \leq \gamma^{PEP}(\bar{x}) = 0$ and according to (i) we can conclude that $\gamma^{DPEP}(\bar{x}) = 0$.

Remark 3.2 *The continuity of the function $F(\bar{x}, \cdot)$ is been used in order to guarantee strong duality between the pair of dual problems $(P^{DPEP}, \bar{x})$ and $(D^{DPEP}, \bar{x})$.*

Remark 3.3 *Notice that the function γ^{PEP} introduced in [11] and the function γ^{DPEP} considered in this paper, are gap function for the equilibrium problem PEP under appropriate assumptions. In general these function do not coincide, even if [11, Theorem 4.1.] and Theorem 3.1 can be applied. For such an example see [11, Example 3.24].*

4 Particular cases

In what follows we particularize problem $(DPEP)$ and we show that we rediscover some gap function for equilibrium problems and variational inequalities considered in the literature in [1, 11].

Let us particularize the dual equilibrium problem $(DPEP)$ to variational inequalities. If we consider for all $x, \bar{x} \in X$ the function $F(x, \bar{x}) = \langle G(x), \bar{x} - x \rangle$ where $G : X \rightarrow X^*$ is a given operator, we rediscover the same framework as in [11, Section 3.3]. The problem $(DPEP)$ becomes: find $\bar{x} \in X$ such that

$$\langle G(x), \bar{x} - x \rangle + \Phi(\bar{x}, 0) \leq \Phi(x, 0), \forall x \in X,$$

which can be rewrite as:

$$\langle G(x), x - \bar{x} \rangle + \Phi(x, 0) - \Phi(\bar{x}, 0) \geq 0, \forall x \in X.$$

In this case

$$\gamma^{DPEP}(\bar{x}) = \inf_{x^* \in X^*, y^* \in Y^*} \left\{ \sup_{x \in X} [\langle x^*, x \rangle + \langle G(x), \bar{x} - x \rangle] + \Phi^*(-x^*, y^*) \right\} + \Phi(\bar{x}, 0),$$

or, equivalently

$$\gamma^{DPEP}(\bar{x}) = \inf_{x^* \in X^*, y^* \in Y^*} \left\{ \sup_{x \in X} [\langle x^*, x \rangle - \langle G(x), x - \bar{x} \rangle] + \Phi^*(-x^*, y^*) \right\} + \Phi(\bar{x}, 0)$$

which is exactly the dual gap function γ'^{Φ} introduced in [11] for the general variational inequality $(VI)^{\Phi}$.

Let us particularize now the perturbation function $\Phi^{f,g} : X \times X \rightarrow \overline{\mathbb{R}}$, $\Phi^{f,g}(x, y) = f(x) + g(x+y)$, where $f, g : X \rightarrow \overline{\mathbb{R}}$ are proper functions fulfilling $\text{dom } f \cap \text{dom } g \neq \emptyset$. Its conjugate function $(\Phi^{f,g})^* : X^* \times X^* \rightarrow \overline{\mathbb{R}}$ is given by (see [7, 22]) $(\Phi^{f,g})^*(x^*, y^*) = f^*(x^* - y^*) + g^*(y^*)$, $\forall (x^*, y^*) \in X^* \times X^*$.

Then, the equilibrium problem (*DPEP*) becomes: find an element $\bar{x} \in X$ such that

$$(DPEP_{f,g}) \quad F(x, \bar{x}) + f(\bar{x}) + g(\bar{x}) \leq f(x) + g(x) \geq 0 \quad \forall x \in X.$$

The function γ^{DPEP} is nothing else than

$$\begin{aligned} \gamma_{f,g}^{DPEP}(\bar{x}) = \inf_{x^* \in X^*, y^* \in Y^*} \left\{ \sup_{x \in X} [\langle x^*, x \rangle + F(x, \bar{x})] + f^*(-x^* - y^*) + \right. \\ \left. + g^*(y^*) \right\} + f(\bar{x}) + g(\bar{x}). \end{aligned}$$

If we further specialize this case to $f = \delta_K$, where $K \subseteq X$ is a nonempty set and $g \equiv 0$, the above function is (since $g^* = \delta_{\{0\}}$)

$$\gamma_{K,0}^{DPEP}(\bar{x}) = \inf_{x^* \in X^*} \left\{ \sup_{x \in X} [\langle x^*, x \rangle + F(x, \bar{x})] + \sigma_K(-x^*) \right\} + \delta_K(\bar{x}),$$

hence $\gamma_{K,0}^{DPEP}(\bar{x}) = \gamma_F^{DEP}(\bar{x})$ for all $\bar{x} \in K$, where γ_F^{DEP} is the function introduced in [1], when considering the equilibrium problem: find an element $\bar{x} \in K$ such that

$$(DEP) \quad -F(x, \bar{x}) \geq 0 \quad \forall x \in K.$$

Remark 4.1 *As the authors showed in [11, Remark 4.2], the continuity of the function $F(\bar{x}, \cdot)$ used in order to guarantee the equality $v(P^{PEP}, \bar{x}) = v(D^{PEP}, \bar{x})$, can be replaced with a closedness type regularity condition and, in analogy, one can give some weak regularity conditions such that γ^{DPEP} becomes a gap function for the equilibrium problem (*PEP*).*

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