

Set-valued solutions of a general linear equation

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ABSTRACT. In this paper we obtain a representation of set-valued solutions of a general linear equation in topological vector spaces

1 Introduction and notations

Let X be a real vector space and Y a real Hausdorff topological vector space. We denote by $P_0(Y)$ the family of all nonempty subsets of Y , by $cc(Y)$ the set of all nonempty convex and compact subsets of Y and by clA the closedness of a set $A \in P_0(Y)$. Let $A, B \in P_0(Y)$ and $\lambda \in \mathbb{R}$. We define the sets $A + B$ and λA by the relations:

$$(1.1) \quad A + B = \{a + b \mid a \in A, b \in B\},$$

$$\lambda A = \{\lambda a \mid a \in A\}.$$

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Then $\lambda(A + B) = \lambda A + \lambda B$ and $(\lambda + \mu)A \subseteq \lambda A + \mu A$ for every $\mu, \lambda \in \mathbb{R}$. If A is a convex set and $\lambda\mu \geq 0$, then

$$(1.2) \quad (\lambda + \mu)A = \lambda A + \mu A.$$

Let $(A_n)_{n \geq 0}$ be a sequence of sets in $P_0(Y)$. The notation $A_n \rightarrow A$ means that the sequence $(A_n)_{n \geq 0}$ converges to A with respect to the Hausdorff topology on $P_0(Y)$. The following results, presented bellow, will be useful in the sequel (see for more details [3, 4, 9, 12]).

Lemma 1.1 *If $(A_n)_{n \geq 0}$ is a decreasing sequence of compact and nonempty subsets of Y , then*

$$A_n \rightarrow \bigcap_{n=1}^{\infty} A_n.$$

Lemma 1.2 *If A is a bounded subset of Y and $(t_n)_{n \geq 0}$ is a real sequence converging to $t \in \mathbb{R}$, then $t_n A \rightarrow tA$.*

Lemma 1.3 *If $(A_n)_{n \geq 0}, (B_n)_{n \geq 0}$ are two sequences of subsets of $\mathcal{P}_o(Y)$ and $A_n \rightarrow A, B_n \rightarrow B$, then $A_n + B_n \rightarrow A + B$.*

Lemma 1.4 *If $(A_n)_{n \geq 0}$ is a sequences of subsets of $\mathcal{P}_o(Y)$ such that $A_n \rightarrow A, A_n \rightarrow B$, then $clA = clB$.*

An important result, which will be used several times in this paper, is Rådström's cancellation law.

Lemma 1.5 *Let $A, B, C \in \mathcal{P}_0(Y)$ such that $A + C \subseteq B + C$. If B is closed and convex and C is bounded, then $A \subseteq B$.*

As a consequence it follows:

Corollary 1.1 *Let $A, B, C \in \mathcal{P}_0(Y)$ such that $A + C = B + C$. If A, B are closed and convex and C is bounded, then $A = B$.*

Let a, b, p, q be real numbers different of zero, $k \in X$ and $C \in P_0(Y)$. In this paper we are looking for set-valued solutions of the general linear equation

$$(1.3) \quad F(ax + by + k) = pF(x) + qF(y) + C,$$

where $F : X \rightarrow P_0(Y)$. The single-valued solutions of equation (1.3) are of the form

$$f(x) = g(x) + c,$$

where $g : X \rightarrow Y$ is an additive function and $c \in Y$ is a constant (see [1, 6]).

Remark also that if C is a convex set $p > 0, q > 0$ and $p + q < 1$, then

$$F(x) = \frac{1}{1 - p - q} C, x \in X,$$

is a constant solution of the equation (1.3). For $a = b = p = q = 1, k = 0_X$ and $C = \{0_Y\}$ in (1.3) we get the Cauchy functional equation. For $a = b = p = q = \frac{1}{2}, k = 0_X$ and $C = \{0_Y\}$ we get the Jensen functional equation. For single-valued maps the general linear equation was studied by Aczél, Daróczy and Losonczi (see [1, 5, 6]). For more results on the characterization and the representation of set-valued solutions of Cauchy, Jensen, Pexider and Quadratic Equation we refer the reader to [2, 7, 8, 9]. A representation of locally compact set-valued solutions of a particular form of equation (1.3), given below

$$(1.4) \quad F(ax + by) = pF(x) + qF(y),$$

$a, b, p, q \in \mathbb{R}, a \cdot b \neq 0, p > 0, q > 0$, was obtained by Popa and Vornicescu in [11] under some conditions satisfied by the coefficients a, b, p, q . The goal of this paper is to obtain a representation of set-valued solutions $F : X \rightarrow cc(Y)$ of the equation (1.3). In this way we complement the results obtained in [11].

2 Main result

We present first two auxiliary results.

Lemma 2.1 *Let $(S, +)$ be a commutative semigroup with zero and C a non-empty, closed, convex and bounded subset of Y with $0_Y \in C$. A set-valued map $F : S \rightarrow cc(Y)$ satisfies the functional equation*

$$(2.1) \quad F(x + y) + C = F(x) + F(y), \quad x, y \in S,$$

if and only if there exists an additive set-valued map $A : S \rightarrow cc(Y)$ such that

$$(2.2) \quad F(x) = A(x) + C, \quad x \in S.$$

Proof. Let $F : S \rightarrow cc(Y)$ be a solution of the equation (2.1). For $x = y$ in (2.1) one gets

$$(2.3) \quad F(2x) + C = 2F(x).$$

We prove by induction that

$$(2.4) \quad F(nx) + (n-1)C = nF(x), \quad x \in X, \quad n \in \mathbb{N}, n \geq 1.$$

For $n = 1$ the relation (2.4) is obvious. Now, suppose that the relation (2.4) holds for some $n \in \mathbb{N}$. We prove that

$$F((n+1)x) + nC = (n+1)F(x).$$

Indeed,

$$\begin{aligned} F((n+1)x) + nC &= F(nx + x) + C + (n-1)C \\ &= F(nx) + F(x) + (n-1)C \\ &= nF(x) + F(x) = (n+1)F(x). \end{aligned}$$

Consequently,

$$F(2^n x) + (2^n - 1)C = 2^n F(x), \quad x \in X, n \in \mathbb{N},$$

or

$$(2.5) \quad \frac{F(2^n x)}{2^n} + \frac{2^n - 1}{2^n} C = F(x), \quad x \in X, n \in \mathbb{N}.$$

Now let us define the sequence of sets $(A_n(x))_{n \geq 0}$ by

$$A_n(x) := \frac{F(2^n x)}{2^n}, \quad \text{for every } n \in \mathbb{N}.$$

We prove that the sequence $(A_n(x))_{n \geq 0}$ is decreasing. Indeed,

$$\begin{aligned} A_{n+1}(x) &= \frac{F(2^{n+1}x)}{2^{n+1}} = \frac{F(2^n x + 2^n x)}{2^{n+1}} \subseteq \frac{F(2^n x + 2^n x) + C}{2^{n+1}} \\ &= \frac{F(2^n x) + F(2^n x)}{2^{n+1}} = A_n(x), \quad n \in \mathbb{N}. \end{aligned}$$

Denoting by $A(x) := \bigcap_{n=0}^{\infty} A_n(x)$ one gets in view of Lemma 1.1 that

$$(2.6) \quad A_n(x) \rightarrow A(x),$$

and $A(x) \in cc(Y)$.

Consider further the sequence $(B_n)_{n \geq 0}$ defined by

$$B_n := \frac{2^n - 1}{2^n} C, \quad \text{for every } n \geq 0.$$

Since C is a bounded set and $\lim_{n \rightarrow \infty} \frac{2^n - 1}{2^n} = 1$ it follows, in view of Lemma (1.2), that $B_n \rightarrow C$.

The relation (2.5) can be rewritten as well in the form

$$(2.7) \quad A_n(x) + B_n = F(x), \quad x \in S, n \geq 0.$$

Now, according to Lemma 1.1 and 1.4 it follows that

$$A_n(x) + B_n \rightarrow A(x) + C,$$

and

$$cl(A(x) + C) = clF(x), \quad x \in S,$$

which entails to

$$A(x) + C = F(x), \quad x \in S,$$

since $A(x) + C$ is closed as a sum of a compact and a closed set. Replacing the above expression of $F(x)$ in (2.1) we get

$$A(x + y) + C + C = A(x) + A(y) + C + C,$$

and according to Radström's cancellation law it follows that A is an additive set-valued map, i.e. $A(x + y) = A(x) + A(y)$ for all $x, y \in S$.

Conversely, if $F(x) = A(x) + C$ where $A : X \rightarrow cc(Y)$ is an additive set-valued map we obtain

$$\begin{aligned} F(x + y) + C &= A(x + y) + C + C = A(x) + C + A(y) + C = \\ &= F(x) + F(y) \quad \text{for all } x, y \in S. \end{aligned}$$

■

Lemma 2.2 *Suppose that $F : X \rightarrow P_0(Y)$ is a solution of the equation (1.3). Then:*

$$(2.8) \quad F(x + y) + F(0_X) = F(x) + F(y), x, y \in X.$$

Proof. Suppose that F satisfies the equation (1.3). For $x = 0_X$ in (1.3) we get

$$F(by + k) = pF(0_X) + qF(y) + C, y \in X,$$

and for $y = -\frac{k}{b}$ in (1.3) it follows

$$F(ax) = pF(x) + qF(-\frac{k}{b}) + C, x \in X.$$

For $x = 0_X$ the previous relation becomes

$$F(0_X) = pF(0_X) + qF(-\frac{k}{b}) + C, x \in X.$$

Let now $u, v \in X$. We obtain:

$$\begin{aligned} F(u + v) + F(0_X) &= F(a\frac{u}{a} + b\frac{v-k}{b} + k) + F(0_X) \\ &= pF(\frac{u}{a}) + qF(\frac{v-k}{b}) + C + F(0_X) \\ &= (pF(\frac{u}{a}) + qF(\frac{v-k}{b}) + C) + (qF(\frac{v-k}{b}) + pF(0_X) + C) \\ &= F(u) + F(v). \end{aligned}$$

■

The main result of the paper is contained in the next theorem.

Theorem 2.1 *Suppose that $F : X \rightarrow cc(Y)$ satisfies the equation (1.3). Then there exists an additive single-valued map $f : X \rightarrow Y$ and a convex and compact set M such that*

$$(2.9) \quad F(x) = f(x) + M, \quad x \in X.$$

Proof. Let $\alpha \in F(0_X)$ and define $G : X \rightarrow cc(Y)$ by $G(x) = F(x) - \{\alpha\}$. Then $0_Y \in G(0_X)$ and G satisfies the relation

$$G(ax + by + k) = pG(x) + qG(y) + D,$$

where $D = (p + q - 1)\alpha + C$. Then, according to Lemma 2.2, G satisfies the relation

$$G(x + y) + G(0_X) = G(x) + G(y), \quad x, y \in X.$$

Taking account of Lemma 2.1 it follows that there exists an additive set-valued map $A : X \rightarrow cc(Y)$ such that

$$G(x) = A(x) + G(0_X), \quad x \in X.$$

The set-valued map A is single-valued since it has bounded values (see [9, 10]). Letting now $f = A, f : X \rightarrow Y$, and $M = F(0_X)$ we get

$$F(x) = f(x) + M, \quad x \in X.$$

■

The theorem 2.1 gives a representation of the solutions of the equation (1.3). A natural question is the following: under what conditions the equation (1.3) admits a solution given by (2.9)? A partial answer to this equation is given in the next theorem.

Theorem 2.2 *Suppose that $a, b \in \mathbb{Q}$. If a nonconstant set-valued map $F : X \rightarrow cc(Y)$ satisfies the equation (1.3), then $p = a$ and $q = b$.*

Proof. Suppose that $F : X \rightarrow cc(Y)$ is a solution of (1.3). Then there exists an additive single-valued map $f : X \rightarrow Y$ such that $F(x) = f(x) + F(0_X)$. Replacing F in the equation (1.3) it follows

$$af(x) + bf(y) + f(k) + F(0_X) = pf(x) + qf(y) + pF(0_X) + qF(0_X) + C$$

for all $x, y \in X$. For $x = y = 0_X$ the previous relation leads to

$$f(k) + F(0_X) = pF(0_X) + qF(0_X) + C,$$

hence

$$(2.10) \quad qf(x) + bf(y) = pf(x) + qf(y), x, y \in X.$$

Since F is a nonconstant set-valued function, there exists $x_0 \in X$ such that $f(x_0) \neq 0_Y$. For $x = x_0$ and $y = 0_X$ in (2.10) we get $a = p$ and for $x = 0_Y$ and $y = 0_X$ we get $b = q$.

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