

Stability and bifurcations in a deformed version of the T system

CRISTIANA CĂPLESCU CRISTIAN LĂZUREANU
(TIMIȘOARA) (TIMIȘOARA)

ABSTRACT. In this paper we consider a deformed version of the T system obtained via integrable deformation method. We discuss the number of the equilibrium points as well as their stability. Moreover, we point out some bifurcations displayed by the considered system.

KEY WORDS: integrable deformation, chaotic system, stability, Hopf bifurcation

MSC 2010: 37C75, 70K20, 70K50

1 Introduction

The study of the stability and bifurcations represents a fundamental part for understanding the dynamics of evolution systems. The systems

[◇]Cristiana Căplescu, Politehnica University of Timișoara, P-ta Victoriei 2, 300 006, Timișoara, Romania
email: cristiana.caplescu@upt.ro

[◇]Cristian Lăzureanu, Politehnica University of Timișoara, P-ta Victoriei 2, 300 006, Timișoara, Romania
email: cristian.lazureanu@upt.ro

obtained by perturbations of the known systems can display the same behavior or not. A particular perturbation is given by the so called integrable deformation [6]. More precisely, if a three-dimensional system of ODEs has two functionally independent constants of motion, then these functions lead to the equations of motion. Therefore, an alteration of the constants of motion gives rise to a new system that has also two constants of motion (see integrable deformations of the Euler top [6]). Using this idea integrable deformations of some three-dimensional Hamilton-Poisson systems were constructed [10, 11, 13]. In [12], this method was extended to arbitrary three-dimensional system of differential equations and applied to some well-known chaotic systems (Lorenz system, Rössler system, and Chen system). Moreover, the idea that new chaotic systems can be obtained by this method was highlighted.

Due to their applications (see, e.g., [1, 3, 4, 16, 22]), more and more systems with chaotic behavior were introduced. Such a system is the T system [17], given by

$$(1.1) \quad \begin{cases} \dot{x} = -ax + ay \\ \dot{y} = (c - a)x - axz \\ \dot{z} = -bz + xy \end{cases} \quad , \quad a, b, c \in \mathbb{R}, a \neq 0.$$

Using the method considered in [12], some deformations of the T system were constructed in [14]. We recall this result.

Theorem 1.1 [14] *Let α, β be arbitrary differentiable functions on $\mathbb{R}^3$. Then an integrable deformation of system (1.1) is given by the following system*

$$(1.2) \quad \begin{cases} \dot{x} = -ax + ay + x(az\alpha_y - y\alpha_z + \alpha_y\beta_z - \alpha_z\beta_y) \\ \dot{y} = (c - a)x - axz - x(\beta_z + az\alpha_x + \alpha_x\beta_z - \alpha_z\beta_x) \\ \dot{z} = -bz + xy + x(\beta_y + y\alpha_x + \alpha_x\beta_y - \alpha_y\beta_x) \end{cases} ,$$

$a, b, c \in \mathbb{R}$, where we denote $f_x := \frac{\partial f}{\partial x}$.

The functions α and β are called deformation functions. If their vanish, then (1.2) becomes the T system.

In this paper we consider the deformation functions given by

$$\alpha(x, y, z) = gz, \quad \beta(x, y, z) = 0,$$

where $g \in \mathbb{R}$ is the deformation parameter. We obtain the following integrable deformation of the T system:

$$(1.3) \quad \begin{cases} \dot{x} = -ax + ay - gxy \\ \dot{y} = (c - a)x - axz \\ \dot{z} = -bz + xy \end{cases}, \quad a, b, c \in \mathbb{R}, a \neq 0, g \in \mathbb{R}.$$

In [14], the case $a = 2, b = 0.2, c = 30$ was considered. The chaotic behavior was pointed out. In this paper we discuss the stability of the equilibrium points of system (1.3). In addition, we verify the possibility that this system to display some bifurcations.

The paper is organized as follows. In Section 2 we study the stability of the particular considered system. The new system has a new parameter g , which is in fact a tuning parameter. The number of equilibrium points of the considered system depends on the values of the parameter g . We notice two bifurcation values of g for which the number of the equilibrium points changes. In Section 3 we analyze conditions in which Hopf bifurcations occur in the dynamics of the considered system. In the final section we give concluding remarks.

2 Stability of a deformed version of the T system

In this section we study the stability of system (1.3). Consider $a, b, c \in (0, \infty)$ and $g \in \mathbb{R}$.

We recall that if $a, b, c \in (0, \infty)$, then the T system has three equilibrium points for $c > a$ and one equilibrium point for $c \leq a$ [17]. In our case, the number of the equilibrium points depends on the parameter g .

Next result gives the equilibrium points of system (1.3).

Proposition 2.1 *Let $a, b, c \in (0, \infty), g \in \mathbb{R}$ and $g_1 = -2\sqrt{\frac{a^3}{ab-bc}}, g_2 =$*

$$2\sqrt{\frac{a^3}{ab-bc}}.$$

i) If $c > a$ or $c < a$ and $g \in (-\infty, g_1) \cup (g_2, \infty)$, then system (1.3) has three

equilibrium points, namely

$$E_0(0, 0, 0), E_1(x_1, -x_2, \frac{c-a}{a}), E_2(x_2, -x_1, \frac{c-a}{a}),$$

where

$$x_1 = \frac{bg(a-c) - \sqrt{b(c-a)(4a^3 - abg^2 + bcg^2)}}{2a^2},$$

$$x_2 = \frac{bg(a-c) + \sqrt{b(c-a)(4a^3 - abg^2 + bcg^2)}}{2a^2}.$$

ii) If $c < a$ and $g = g_1$ or $g = g_2$, then system (1.3) has two equilibrium points, E_0 and E_1^- or E_1^+ , where

$$E_1^-(-\sqrt{\frac{b(a-c)}{a}}, \sqrt{\frac{b(a-c)}{a}}, \frac{c-a}{a}), E_1^+(\sqrt{\frac{b(a-c)}{a}}, -\sqrt{\frac{b(a-c)}{a}}, \frac{c-a}{a}).$$

iii) If $a = c$ or $c < a$ and $g \in (g_1, g_2)$, then system (1.3) has only one equilibrium point, namely E_0 .

Remark 2.1 Fixing $a, b, c \in (0, \infty)$ such that $c < a$, the number of equilibrium points changes at the bifurcation points $g = g_1$ and $g = g_2$, while T system has only one equilibrium point.

In the following we give some stability results of the above-mentioned equilibrium points.

In order to study the stability of the equilibrium point E_0 , we have used the same technique as in [17]. We have obtain that the stability of E_0 doesn't depend on g and the result is the same.

Proposition 2.2 Let $a, b, c \in (0, \infty), g \in \mathbb{R}$.

- i) If $c \leq a$, then the equilibrium point $E_0(0, 0, 0)$ is asymptotically stable;
- ii) If $c > a$, then $E_0(0, 0, 0)$ is an unstable equilibrium point.

In the following we study the stability of other equilibrium points.

First, we need to consider the following equation, which is obtained in

the proof of the next proposition:

$$\begin{aligned}
 (2.1) \quad & -4a^{11} + 4a^{10}c - 4a^9bc - a^9c^2 + 2a^8bc^2 - a^7b^2c^2 + (9a^9b - 4a^8b^2 - \\
 & -4a^7b^3 - 18a^8bc + 9a^7b^2c + 8a^6b^3c + 9a^7bc^2 - 5a^6b^2c^2 - 4a^5b^3c^2) \mu^2 \\
 & + (-6a^7b^2 + 5a^6b^3 + a^5b^4 + 18a^6b^2c - 15a^5b^3c - 3a^4b^4c - 18a^5b^2c^2 + 15a^4b^3c^2 \\
 & + 3a^3b^4c^2 + 6a^4b^2c^3 - 5a^3b^3c^3 - a^2b^4c^3) \mu^4 \\
 & + (a^5b^3 - a^4b^4 - 4a^4b^3c + 4a^3b^4c + 6a^3b^3c^2 - 6a^2b^4c^2 - 4a^2b^3c^3 + 4ab^4c^3 \\
 & + ab^3c^4 - b^4c^4) \mu^6 = 0.
 \end{aligned}$$

The real roots of this equation are denoted by $g_1^*, g_2^*, g_3^*, g_4^*$ and so on, and they are increasing ordered provided that exists.

Using MATHEMATICA software, we obtain the following results.

Proposition 2.3 *Let $a, b, c \in (0, \infty), g \in \mathbb{R}$, such that $c > a$.*

- i) E_1 is asymptotically stable iff $a \leq b$ or $a > b, c < \frac{2a^2}{a-b}, g < g_2^*$ or $a > b, c \geq \frac{2a^2}{a-b}, g < g_1^*$;
- ii) E_2 is asymptotically stable iff $a \leq b$ or $a > b, c \leq \frac{2a^2}{a-b}, g > g_1^*$ or $a > b, c > \frac{2a^2}{a-b}, g > g_2^*$.

Proof. i) According to Routh-Hurwitz Theorem [7], a polynomial

$$P(\lambda) = \lambda^3 + a_1\lambda^2 + a_2\lambda + a_3$$

with real coefficients has all roots with strictly negative real part if and only if

$$a_1 > 0, a_3 > 0, a_1a_2 - a_3 > 0.$$

In our case, the characteristic polynomial of the linearized of system (1.3)

at E_1 has the coefficients

$$(2.2) \quad \begin{aligned} a_1 &= \frac{2a^4 + 2a^3b - a^2bg^2 + abcg^2 - ag\Delta}{2a^3}, \\ a_2 &= \frac{2a^3bc - ab^2cg^2 + b^2c^2g^2 - 2abg\Delta + bcg\Delta}{2a^3}, \\ a_3 &= \frac{-4a^5b + 4a^4bc + a^3b^2g^2 - 2a^2b^2cg^2 + ab^2c^2g^2 - a^2bg\Delta + abcg\Delta}{2a^3}. \end{aligned}$$

where $\Delta = \sqrt{b(-a+c)(4a^3 - abg^2 + bcg^2)}$. We get $a_1 > 0, a_2 > 0, a_3 > 0$. Moreover, $a_1a_2 - a_3 > 0$ iff the conditions given by hypothesis hold. Then the conclusion follows. We mention that equation (2.1) represents in fact the condition $a_1a_2 - a_3 = 0$.

ii) Analogously. ■

Remark 2.2 *In order to highlight the dependence of the stability of the equilibrium points E_1 and E_2 by the real parameter g , we gather the stability results in Table 1. Mention that $g_1^* = g_2^* = 0$ in the case $c = \frac{2a^2}{a-b}$. The abbreviations AS and U signify asymptotically stable and unstable, respectively.*

E_1	g		
	$(-\infty, g_1^*)$	(g_1^*, g_2^*)	(g_2^*, ∞)
$c < \frac{2a^2}{a-b}$	AS	AS	U
$c > \frac{2a^2}{a-b}$	AS	U	U
E_2	g		
	$(-\infty, g_1^*)$	(g_1^*, g_2^*)	(g_2^*, ∞)
$c < \frac{2a^2}{a-b}$	U	AS	AS
$c > \frac{2a^2}{a-b}$	U	U	AS

Table 1: Stability of the equilibrium points E_1 and E_2 in the case $c > a > b$.

In the same manner as in the previous proposition we have obtained the next result.

Proposition 2.4 *Let $a, b, c \in (0, \infty), g \in \mathbb{R}$, such that $c < a$.*

1. *Let $g < -2\sqrt{\frac{a^3}{ab-bc}}$. Then*
 - i) *E_1 is asymptotically stable iff $\frac{2b}{3} \leq a < b$, $g < g_1^*$ or $a < \frac{2b}{3}$, $c < \frac{2a^2-3ab}{2(a-b)} - \frac{1}{2}\sqrt{\frac{4a^4+a^2b^2}{(a-b)^2}}$, $g < g_2^*$ or $a < \frac{2b}{3}$, $c \geq \frac{2a^2-3ab}{2(a-b)} - \frac{1}{2}\sqrt{\frac{4a^4+a^2b^2}{(a-b)^2}}$, $g < g_1^*$;*
 - ii) *E_2 is unstable.*
2. *Let $g > 2\sqrt{\frac{a^3}{ab-bc}}$. Then*
 - i) *E_1 is unstable.*
 - ii) *E_2 is asymptotically stable iff $\frac{2b}{3} \leq a < b$, $g > g_4^*$ or $a < \frac{2b}{3}$, $c \leq \frac{2a^2-3ab}{2(a-b)} - \frac{1}{2}\sqrt{\frac{4a^4+a^2b^2}{(a-b)^2}}$, $g > g_3^*$ or $a < \frac{2b}{3}$, $c > \frac{2a^2-3ab}{2(a-b)} - \frac{1}{2}\sqrt{\frac{4a^4+a^2b^2}{(a-b)^2}}$, $g > g_4^*$.*
3. *Let $g = -2\sqrt{\frac{a^3}{ab-bc}}$. Then the equilibrium point E_1^- is unstable.*
4. *Let $g = 2\sqrt{\frac{a^3}{ab-bc}}$. Then the equilibrium point E_1^+ is unstable.*

3 Bifurcations

In this section we present some bifurcations that occur in the dynamics of the considered system.

In [8, 21] the Hopf bifurcations in T system was analyzed. We recall the results, namely T system cannot have a Hopf bifurcation at $O(0, 0, 0)$ and it undergoes a Hopf bifurcation at the equilibrium point E_1 (E_2) when $c = \frac{2a^2}{a-b}$ ($0 < b < a < c$).

We note that system (1.3) can be viewed as the T system with the parametric control $u(x, y, z) = -gxy$, where g is a tuning parameter (see, e.g., [4]). Therefore we will consider g as the bifurcation parameter and we fix the other parameters such that $c > a > b > 0$. Following the proof of Proposition 2.3, the equilibrium points E_1 and E_2 are hyperbolic points, generally. Moreover, the eigenvalues of the linearized of our system at E_1 and E_2 cannot vanish, hence a saddle-node bifurcation is not possible. In the sequel we prove that system (1.3) displays Hopf bifurcations. The transition between asymptotically stable states and unstable states given in Table 1

suggests that g_1^* and g_2^* are critical values of bifurcation of the parameter g , considering a, b, c fixed.

Proposition 3.1 *Let $c > a > b$ fixed and $g \in \mathbb{R}$.*

- i) If $c < \frac{2a^2}{a-b}$, then for $g = g_1^*$ and $g = g_2^*$ system (1.3) displays a Hopf bifurcation at the point E_2 and E_1 respectively.*
- ii) If $c > \frac{2a^2}{a-b}$, then for $g = g_1^*$ and $g = g_2^*$ system (1.3) displays a Hopf bifurcation at the point E_1 and E_2 respectively.*

Proof. Let $g = g_2^*$. Then $a_1 a_2 - a_3 = 0$, where a_1, a_2, a_3 are functions that depend on g given by (2.2). Hence the characteristic polynomial $P(\lambda, g) = \lambda^3 + a_1 \lambda^2 + a_2 \lambda + a_3$ of the linearized of system (1.3) at E_1 reads

$$P(\lambda, g_2^*) = (\lambda + a_1(g_2^*))(\lambda^2 + a_2(g_2^*)),$$

that is its roots are $\lambda_1 = -a_1(g_2^*) < 0$, $\lambda_{2,3} = \pm i\sqrt{a_2(g_2^*)}$. Moreover, if $g < g_2^*$, then $a_1 a_2 - a_3 > 0$ and if $g > g_2^*$, then $a_1 a_2 - a_3 < 0$. Because $a_1, a_2, a_3 > 0$, for all $g \in \mathbb{R}$, it follows that $\lambda_1 = \lambda_1(g) < 0$, for each g . It only remains to prove that in a neighborhood of $g = g_2^*$ the eigenvalues $\lambda_{2,3}(g)$ are not reals.

Let $\lambda(g) = u(g) + iv(g)$ such that $u(g_2^*) = 0$ and $v(g_2^*) = \sqrt{a_2(g_2^*)}$. Using the Implicit Function Theorem we obtain that there is a neighborhood U of $g = g_2^*$ such that the equation $(\operatorname{Re}(P(\lambda, g)), \operatorname{Im}(P(\lambda, g))) = (0, 0)$ defines a continuously differentiable function $(u(g), v(g))$ and

$$\begin{aligned} u'(g_2^*) &= \frac{a_3'(g_2^*) - a_1'(g_2^*)a_2(g_2^*) - a_1(g_2^*)a_2'(g_2^*)}{2a_1^2(g_2^*) + 2a_2(g_2^*)}, \\ v'(g_2^*) &= \frac{a_1(g_2^*)a_3'(g_2^*) + a_2(g_2^*)a_2'(g_2^*) - a_1(g_2^*)a_2(g_2^*)a_1'(g_2^*)}{2a_1^2(g_2^*) + 2a_2(g_2^*)}, \end{aligned}$$

where $w'(g_2^*) = \frac{dw}{dg}(g_2^*)$. In our case we generally get $u'(g_2^*) \neq 0, v'(g_2^*) \neq 0$, hence u and v do not vanish on $U \setminus \{g_2^*\}$. Therefore our system displays a Hopf bifurcation at the point E_1 .

The other cases follow analogously. ■

Remark 3.1 *Let a, b be fixed. In the light of the Hopf bifurcation result of T system, the above result tells us that for every c there is a value of g such*

that the considered deformation of system T undergoes a Hopf bifurcation. Moreover, if $c = \frac{2a^2}{a-b}$, then $g = 0$, that is we obtain the above-mentioned Hopf bifurcation of the T system.

Remark 3.2 In order to establish if the Hopf bifurcation is nedeenerate, the computation of the value of the first Lyapunov coefficient at $g = g^*$ is necessary (see, e.g., [9]).

To emphasize the Hopf bifurcation, we perform a numerical simulation using MATCONT software. We consider $a = 2, b = 1, c = 3$, hence $c < \frac{2a^2}{a-b}$, and we obtain $g_1^* = -2$ and $g_2^* = 2$. Then for $g = g_2^*$ the eigenvalues of the equilibrium point $E_1(-1, -0.5, 0.5)$ are $\lambda_1 = -2, \lambda_{2,3} = \pm 1.73205$ and the first Lyapunov coefficient is $l_1(E_1) = -0.1449275$. Therefore a supercritical Hopf bifurcation is pointed out.

4 Conclusions

In this paper we analyze a deformed version of the T system. The new parameter g introduced in the system leads to a discussion on the number of equilibrium points and their stability. In addition, we establish that Hopf bifurcations occur in the considered system for any c , when g varies.

Further studies of the considered system regarding dynamics, complexity, control, and synchronization are possible (see, e.g., [2, 4, 5, 15, 18, 19, 20]).

5 ACKNOWLEDGMENTS

This research was partially supported by Horizon2020-2017-RISE-777911 project.

References

- [1] A. Babloyantz, C. Nicolis, and J. M. Salazar. Evidence of chaotic dynamics of brain activity during the sleep cycle. *Phys. Lett. A*, 111: 152–156, 1985.

- [2] T. Bînzar, C. Lăzureanu. On a new chaotic system. *Math. Method Appl. Sci.*, 38(8): 1631–1641, 2015.
- [3] G. Chen, X. Dong. *From Chaos to Order: Methodologies, Perspectives and Applications*. World Scientific Pub. Co., Singapore, 1998.
- [4] G. Chen. *Controlling Chaos and Bifurcations in Engineering Systems*. CRC Press, Boca Raton, FL, 1999.
- [5] Y. Chen, Z.-Y. Yan. Chaos Control in a New Three-Dimensional Chaotic T System. *Commun. Theor. Phys.*, 49(4): 951–954, 2008.
- [6] A. Galajinsky. Remark on integrable deformations of the Euler top. *J. Math. Anal. Appl.*, 416(2): 995–997, 2014.
- [7] F. R. Gantmacher. *Matrix Theory*, Vol. II. Chelsea Pub. Co., New York, 2000.
- [8] B. Jiang, X. Han, and Q. Bi. Hopf bifurcation analysis in the T system. *Nonlinear Anal.-Real*, 11(1): 522–527, 2010.
- [9] Y. A. Kuznetsov,. *Elements of applied bifurcation theory*. Springer-Verlag, New York, 1998.
- [10] C. Lăzureanu. On the Hamilton-Poisson realizations of the integrable deformations of the Maxwell-Bloch equations. *Cr. Acad. Sci. Math.*, 355(5): 596–600, 2017.
- [11] C. Lăzureanu. Hamilton-Poisson Realizations of the Integrable Deformations of the Rikitake System. *Adv. Math. Phys.* 2017: Article ID 4596951, 9 pages, 2017.
- [12] C. Lăzureanu. Integrable Deformations of Three-Dimensional Chaotic Systems. *Int. J. Bifurcat. Chaos*, 28(5): Article ID 1850066, 7 pages, 2018.
- [13] C. Lăzureanu, C. Petrişor. Stability and Energy-Casimir Mapping for Integrable Deformations of the Kermack-McKendrick System. *Adv. Math. Phys.*, 2018: Article ID 5398768, 9 pages, 2018.

- [14] C. Lăzureanu, C. Căplescu. On a deformed version of the T system. Proceedings of The XV-th Int. Conf. on Mathematics and its Applications, Timișoara, 2019, 90–97.
- [15] L. M. Pecora, T. L. Carroll. Synchronization in chaotic systems. *Phys. Rev. Lett.*, 64: 821–824, 1991.
- [16] M. I. Rabinovich, H. D. I. Abarbanel. The role of chaos in neural systems. *Neuroscience*, 87: 5–14, 1998.
- [17] Gh. Tigan. Analysis of a dynamical system derived from the Lorenz system. *Sci. Bull. Politehnica University of Timisoara*, 50(64), Fascicola 1: 61–72, 2005.
- [18] G. Tigan, D. Opris. Analysis of a 3D chaotic system. *Chaos Soliton. Fract.*, 36: 1315–1319, 2008.
- [19] G. Tigan, D. Constantinescu. Heteroclinic orbits in the T and the Lü systems. *Chaos Soliton. Fract.*, 42(1): 20–23, 2009.
- [20] G. Tigan, J. Llibre. Heteroclinic, Homoclinic and Closed Orbits in the Chen System. *Int. J. Bifurcat. Chaos*, 26(4): Article ID 1650072, 6 pages, 2016.
- [21] R. A. Van Gorder, S. Roy Choudhury. Analytical Hopf Bifurcation and Stability Analysis of T System. *Commun. Theor. Phys.*, 55(4): 609–616, 2011.
- [22] T. Yang, L. O. Chua. Impulsive stabilization for control and synchronization of chaotic systems: theory and application to secure communication. *IEEE T. Circuits-I*, 44(10): 976–988, 1997.