

**Differential sandwich theorem for analytic
functions defined by a generalized multiplier
transformations**

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ABSTRACT. In the present paper we introduce certain differential subordination and superordination properties for analytic functions involving a generalized multiplier transformation which include as special cases the Ruscheweyh derivative operator and the generalized Sălăgean operator introduced by Al-Oboudi. Applying the techniques of differential subordination and superordination we also establish differential sandwich-type theorems.

KEY WORDS: multiplier transformations, differential subordination, differential superordination

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1 Introduction

Let $\mathcal{H} = \mathcal{H}(U)$ denote the class of functions analytic in the unit disc

$$U = \{z \in \mathbf{C} : |z| < 1\}.$$

For $a \in \mathbf{C}$ and $n \in \mathbf{N}$ let $\mathcal{H}[a, n]$ be the subclass of $\mathcal{H}(U)$ consisting of functions of the form

$$f(z) = a + a_n z^n + a_{n+1} z^{n+1} + \dots.$$

Let

$$\mathcal{A}_n = \{f \in \mathcal{H}(U), f(z) = z + a_{n+1} z^{n+1} + \dots\}$$

with $\mathcal{A}_1 := \mathcal{A}$.

We denote by Q the set of functions f that are analytic and injective on $\overline{U} \setminus E(f)$, where

$$E(f) = \left\{ \zeta \in \partial U : \lim_{z \rightarrow \zeta} f(z) = \infty \right\}$$

and are such that $f'(\zeta) \neq 0$ for $\zeta \in \partial U \setminus E(f)$.

Since we use the terms of subordination and superordination, we review here those definitions.

Let $f, F \in \mathcal{H}$. The function f is said to be subordinate to F or F is said to be superordinate to f , if there exists a function w analytic in U , with $w(0) = 0$ and $|w(z)| < 1$, and such that $f(z) = F(w(z))$. In such a case we write $f \prec F$ or $f(z) \prec F(z)$. If F is univalent, then $f \prec F$ if and only if $f(0) = F(0)$ and $f(U) \subset F(U)$.

Since most of the functions considered in this paper and conditions on them are defined uniformly in the unit disk U , we shall omit the requirement " $z \in U$ ".

Let $\psi : \mathbf{C}^3 \times \overline{U} \rightarrow \mathbf{C}$, let h be univalent in U and $q \in Q$. In [7] the authors considered the problem of determining conditions on admissible function ψ such that

$$(1.1) \quad \psi(p(z), zp'(z), z^2 p''(z); z) \prec h(z)$$

implies $p(z) \prec q(z)$, for all functions $p \in \mathcal{H}[a, n]$ that satisfy the differential subordination (1.1).

Moreover, they found conditions so that the function q is the "smallest" function with this property, called the best dominant of the subordination (1.1).

Let $\varphi : \mathbf{C}^3 \times \overline{U} \rightarrow \mathbf{C}$, let $h \in \mathcal{H}$ and $q \in \mathcal{H}[a, n]$. Recently, in [8] the authors studied the dual problem and determined conditions on φ such that

$$(1.2) \quad h(z) \prec \varphi(p(z), zp'(z), z^2p''(z); z)$$

implies $q(z) \prec p(z)$, for all functions $p \in \mathcal{Q}$ that satisfy the above differential superordination.

Moreover, they found conditions so that the function q is the "largest" function with this property, called the best subordinant of the superordination (1.2).

For two functions

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n \quad \text{and} \quad g(z) = z + \sum_{n=2}^{\infty} b_n z^n,$$

the Hadamard product of f and g is defined by

$$(f * g)(z) := z + \sum_{n=2}^{\infty} a_n b_n z^n.$$

2 Preliminaries

We begin our investigation by recalling here a generalized differential operator defined in [5].

Definition 2.1 *Let the function f be in the class $\mathcal{A}_n$. For $m, \beta \in \mathbf{N}_0 = \{0, 1, 2, \dots\}$, $\lambda \geq 0$, $l \geq 0$, we define the following differential operator*

$$(2.1) \quad I^m(\lambda, \beta, l)f(z) := z + \sum_{k=n+1}^{\infty} \left[\frac{1 + \lambda(k-1) + l}{1+l} \right]^m C(\beta, k) a_k z^k$$

where

$$C(\beta, k) := \binom{k + \beta - 1}{\beta} = \frac{(\beta + 1)_{k-1}}{(k-1)!}$$

and

$$(a)_n := \begin{cases} 1, & n = 0 \\ a(a+1) \dots (a+n-1), & n \in \mathbf{N} = \mathbf{N}_0 \setminus \{0\} \end{cases}$$

is Pochhammer symbol.

Using simple computation one obtains the next result.

Proposition 2.1 For $m, \beta \in \mathbf{N}_0$, $\lambda \geq 0$, $l \geq 0$ we have

$$(2.2) \quad z(I^m(\lambda, \beta, l)f(z))' = (1 + \beta)I^m(\lambda, \beta + 1, l)f(z) - \beta I^m(\lambda, \beta, l)f(z).$$

Remark 2.1 Special cases of this operator includes the Ruscheweyh derivative operator denoted by $I^0(1, \beta, 0)f(z) \equiv D_\beta$ defined in [9], the Sălăgean derivative operator $I^m(1, 0, 0)f(z) \equiv D^m$, studied in [10], the generalized Sălăgean operator $I^m(\lambda, 0, 0) \equiv D_\lambda^m$ introduced by Al-Oboudi in [1], the generalized Ruscheweyh derivative operator $I^1(\lambda, \beta, 0)f(z) \equiv D_{\lambda, \beta}$ introduced in [6], the operator $I^m(\lambda, \beta, 0) \equiv D_{\lambda, \beta}^m$ introduced by K. Al-Shaqsi and M. Darus in [2] and finally the operator $I^m(\lambda, 0, l) \equiv I_1(m, \lambda, l)$ introduced in [4].

In this paper, we will derive several subordination and superordination results involving the operator $I^m(\lambda, \beta, l)$. In order to prove our main results, we also need the following results.

Lemma 2.1 [7] Let q be univalent in U , $\gamma \in \mathbf{C}^*$ and suppose

$$\Re \left[1 + \frac{zq''(z)}{q'(z)} \right] > \max \left\{ 0, -\Re \frac{1}{\gamma} \right\}.$$

If p is analytic in U , with $p(0) = q(0)$ and

$$p(z) + \gamma zp'(z) \prec q(z) + \gamma zq'(z),$$

then $p \prec q$, and q is the best dominant.

Lemma 2.2 [3] *Let q be convex in U , $q(0) = a$ and $\gamma \in \mathbf{C}$, $\Re \gamma > 0$.*

If $p \in \mathcal{H}[a, 1] \cap Q$, the function $p(z) + \gamma zp'(z)$ is univalent in U and

$$q(z) + \gamma zq'(z) \prec p(z) + \gamma zp'(z)$$

then

$$q(z) \prec p(z)$$

and q is the best subdominant.

3 Main results

Theorem 3.1 *Let q be univalent in U with $q(0) = 1$, $\alpha \in \mathbf{C}^* = \mathbf{C} \setminus \{0\}$, $m, \beta \in \mathbf{N}_0 = \{0, 1, 2, \dots\}$ and suppose*

$$\Re \left\{ 1 + \frac{zq''(z)}{q'(z)} \right\} > \max \left\{ 0, -\Re \frac{\beta + 1}{\alpha} \right\}.$$

If $f \in \mathcal{A}_n$ satisfies the subordination

$$(3.1) \quad (1 - \alpha) \cdot \frac{I^m(\lambda, \beta, l)f(z)}{z} + \alpha \cdot \frac{I^m(\lambda, \beta + 1, l)f(z)}{z} \prec q(z) + \frac{\alpha}{\beta + 1} zq'(z)$$

then

$$\frac{I^m(\lambda, \beta, l)f(z)}{z} \prec q(z)$$

and q is the best dominant.

Proof. We define the function

$$(3.2) \quad p(z) := \frac{I^m(\lambda, \beta, l)f(z)}{z}.$$

Differentiating (3.2) with respect to z and using the identity (2.2) in the resulting equation we have

$$\frac{zp'(z)}{p(z)} = (\beta + 1) \left(\frac{I^m(\lambda, \beta + 1, l)}{I^m(\lambda, \beta, l)} - 1 \right).$$

Therefore, one obtains

$$(1 - \alpha) \cdot \frac{I^m(\lambda, \beta, l)f(z)}{z} + \alpha \cdot \frac{I^m(\lambda, \beta + 1, l)f(z)}{z} = p(z) + \frac{\alpha}{\beta + 1} zp'(z).$$

The subordination (3.1) from the hypothesis becomes

$$p(z) + \frac{\alpha}{\beta + 1} zp'(z) \prec q(z) + \frac{\alpha}{\beta + 1} zq'(z).$$

We apply now Lemma 2.1 with $\gamma = \frac{\alpha}{\beta + 1}$ to obtain the conclusion of our theorem.

If we consider $m = 0$ and $\beta = 0$ in Theorem 3.1, we obtain the following result.

Corollary 3.1 *Let q be univalent in U with $q(0) = 1$, $\alpha \in \mathbf{C}^* = \mathbf{C} \setminus \{0\}$ and suppose*

$$\Re \left\{ 1 + \frac{zq''(z)}{q'(z)} \right\} > \max \left\{ 0, -\Re \frac{1}{\alpha} \right\}.$$

If $f \in \mathcal{A}_n$ satisfies the subordination

$$(3.3) \quad (1 - \alpha) \cdot \frac{f(z)}{z} + \alpha \cdot f'(z) \prec q(z) + \alpha zq'(z)$$

then

$$\frac{f(z)}{z} \prec q(z)$$

and q is the best dominant of (3.3).

We consider a particular convex function $q(z) = \frac{1 + Az}{1 + Bz}$ to give the following application to Theorem 3.1.

Corollary 3.2 *Let $\alpha \in \mathbf{C}$, $A \neq B$ be such that $-1 \leq B < A \leq 1$ and $\Re \alpha > 0$. If $f \in \mathcal{A}_n$ satisfies the subordination*

$$(3.4) \quad (1 - \alpha) \frac{I^m(\lambda, \beta, l)f(z)}{z} + \alpha \frac{I^m(\lambda, \beta + 1, l)f(z)}{z} \prec \frac{1 + Az}{1 + Bz} + \frac{\alpha}{\beta + 1} \frac{(A - B)z}{(1 + Bz)^2}$$

then

$$\frac{I^m(\lambda, \beta, l)f(z)}{z} \prec \frac{1 + Az}{1 + Bz}$$

and $q(z) = \frac{1 + Az}{1 + Bz}$ is the best dominant of (3.4).

Corollary 3.3 Let $\alpha \in \mathbf{C}$ and $\Re \alpha > 0$. If $f \in \mathcal{A}_n$ satisfies the subordination

$$(3.5) \quad (1-\alpha) \frac{I^m(\lambda, \beta, l)f(z)}{z} + \alpha \frac{I^m(\lambda, \beta + 1, l)f(z)}{z} \prec \frac{1+z}{1-z} + \frac{\alpha}{\beta+1} \frac{2z}{(1-z)^2}$$

$$\frac{I^m(\lambda, \beta, l)f(z)}{z} \prec \frac{1+z}{1-z}$$

and $q(z) = \frac{1+z}{1-z}$ is the best dominant of (3.5).

Theorem 3.2 Let q be convex in U with $q(0) = 1$, $\alpha \in \mathbf{C}$, $\Re \alpha > 0$. If $f \in \mathcal{A}_n$ such that $\frac{I^m(\lambda, \beta, l)f(z)}{z} \in \mathcal{H}[1, 1] \cap Q$ and

$$(1-\alpha) \cdot \frac{I^m(\lambda, \beta, l)f(z)}{z} + \alpha \cdot \frac{I^m(\lambda, \beta + 1, l)f(z)}{z}$$

is univalent in U and satisfies the superordination

$$(3.6) \quad q(z) + \frac{\alpha}{\beta+1} zq'(z) \prec (1-\alpha) \cdot \frac{I^m(\lambda, \beta, l)f(z)}{z} + \alpha \cdot \frac{I^m(\lambda, \beta + 1, l)f(z)}{z},$$

then

$$q(z) \prec \frac{I^m(\lambda, \beta, l)f(z)}{z}$$

and q is the best subdominant.

Proof. Let $p(z)$ be given by (3.2) and proceeding as in the proof of Theorem 3.1 the subordination (3.6) becomes

$$q(z) + \frac{\alpha}{\beta+1} zq'(z) \prec p(z) + \frac{\alpha}{\beta+1} zp'(z).$$

The conclusion of this theorem follows by applying Lemma 2.2.

Taking $m = 0$ and $\beta = 0$ in Theorem 3.2, we obtain the following result.

Corollary 3.4 *Let q be convex in U , with $q(0) = 1$, $\alpha \in \mathbf{C}$, with $\Re \alpha > 0$. If $f \in \mathcal{A}_n$ such that $\frac{f(z)}{z} \in \mathcal{H}[1, 1] \cap Q$ and*

$$(1 - \alpha) \cdot \frac{f(z)}{z} + \alpha \cdot f'(z)$$

is univalent in U and satisfies the superordination

$$q(z) + \alpha z q'(z) \prec (1 - \alpha) \cdot \frac{f(z)}{z} + \alpha \cdot f'(z),$$

then

$$q(z) \prec \frac{f(z)}{z}$$

and q is the best subdominant.

Corollary 3.5 *Let $\alpha \in \mathbf{C}$, $A \neq B$ be such that $-1 \leq B < A \leq 1$ and $\Re \alpha > 0$. If $f \in \mathcal{A}_n$ such that $\frac{I^m(\lambda, \beta, l)f(z)}{z} \in \mathcal{H}[1, 1] \cap Q$ and*

$$(1 - \alpha) \frac{I^m(\lambda, \beta, l)f(z)}{z} + \alpha \frac{I^m(\lambda, \beta + 1, l)f(z)}{z}$$

is univalent in U and satisfies the superordination

$$\frac{1 + Az}{1 + Bz} + \frac{\alpha}{\beta + 1} \frac{(A - B)z}{(1 + Bz)^2} \prec (1 - \alpha) \frac{I^m(\lambda, \beta, l)f(z)}{z} + \alpha \frac{I^m(\lambda, \beta + 1, l)f(z)}{z}$$

then

$$\frac{1 + Az}{1 + Bz} \prec \frac{I^m(\lambda, \beta, l)f(z)}{z}$$

and $q(z) = \frac{1 + Az}{1 + Bz}$ is the best subdominant.

Corollary 3.6 *Let $\alpha \in \mathbf{C}$, $\Re \alpha > 0$. If $f \in \mathcal{A}_n$ such that*

$$\frac{I^m(\lambda, \beta, l)f(z)}{z} \in \mathcal{H}[1, 1] \cap Q$$

and

$$(1 - \alpha) \frac{I^m(\lambda, \beta, l)f(z)}{z} + \alpha \frac{I^m(\lambda, \beta + 1, l)f(z)}{z}$$

is univalent in U and satisfies the superordination

$$\frac{1+z}{1-z} + \frac{\alpha}{\beta+1} \frac{2z}{(1-z)^2} \prec (1 - \alpha) \frac{I^m(\lambda, \beta, l)f(z)}{z} + \alpha \frac{I^m(\lambda, \beta + 1, l)f(z)}{z}$$

then

$$\frac{1+z}{1-z} \prec \frac{I^m(\lambda, \beta, l)f(z)}{z}$$

and $q(z) = \frac{1+z}{1-z}$ is the best subdominant.

Combining the results of Theorem 3.1 and Theorem 3.2 we state the following sandwich result.

Theorem 3.3 *Let q_1, q_2 be convex in U with $q_1(0) = q_2(0) = 1$, $\alpha \in \mathbf{C}$, $\Re \alpha > 0$ and q_2 satisfies the inequality*

$$\Re \left\{ 1 + \frac{z q_2''(z)}{q_2'(z)} \right\} > \max \left\{ 0, -\Re \frac{\alpha}{\beta+1} \right\}.$$

If $f \in \mathcal{A}_n$ such that $\frac{I^m(\lambda, \beta, l)f(z)}{z} \in \mathcal{H}[1, 1] \cap \mathcal{Q}$ and

$$(1 - \alpha) \frac{I^m(\lambda, \beta, l)f(z)}{z} + \alpha \frac{I^m(\lambda, \beta + 1, l)f(z)}{z}$$

is univalent in U and satisfies

$$\begin{aligned} q_1(z) + \frac{\alpha}{\beta+1} z q_1'(z) &\prec (1 - \alpha) \cdot \frac{I^m(\lambda, \beta, l)f(z)}{z} + \alpha \cdot \frac{I^m(\lambda, \beta + 1, l)f(z)}{z} \\ &\prec q_2(z) + \frac{\alpha}{\beta+1} z q_2'(z), \end{aligned}$$

then

$$q_1(z) \prec \frac{I^m(\lambda, \beta, l)f(z)}{z} \prec q_2(z)$$

and q_1, q_2 are the best subdominant and the best dominant respectively.

Combining also Corollary 3.1 and Corollary 3.4 we obtaining the corresponding sandwich results for the operator $\frac{f(z)}{z}$.

Corollary 3.7 *Let q_1, q_2 be convex in U with $q_1(0) = q_2(0) = 1$, $\alpha \in \mathbf{C}$, $\Re \alpha > 0$ and q_2 satisfies the inequality*

$$\Re \left\{ 1 + \frac{z q_2''(z)}{q_2'(z)} \right\} > \max \{0, -\Re \alpha\}.$$

If $f \in \mathcal{A}_n$ such that $\frac{f(z)}{z} \in \mathcal{H}[1, 1] \cap Q$ and

$$(1 - \alpha) \cdot \frac{f(z)}{z} + \alpha \cdot f'(z)$$

is univalent in U and satisfies

$$q_1(z) + \alpha z q_1'(z) \prec (1 - \alpha) \cdot \frac{f(z)}{z} + \alpha \cdot f'(z) \prec q_2(z) + \alpha z q_2'(z)$$

then

$$q_1(z) \prec \frac{f(z)}{z} \prec q_2(z)$$

and q_1, q_2 are the best subordinant and the best dominant respectively.

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