

On some inequalities using h -convexity of functions

DANIEL IANOȘI

(CLUJ-NAPOCA)

ABSTRACT. We establish some inequalities of the Ostrowski type involving left-sided and right-sided ψ -Riemann-Liouville fractional integrals for convex functions. We give some applications using the Peano representation formula for linear functionals.

KEY WORDS: Ostrowski's inequality, ψ -Riemann-Liouville fractional integrals, Peano's Theorem, h -convexity

1 Introduction and preliminaries

In 1938 A. Ostrowski proved the following theorem.

Theorem 1.1 (see [6]) *Let $f : I \rightarrow \mathbb{R}$, where $I \subset \mathbb{R}$ is an interval, be a differentiable mapping in the interior of I . If $|f'(t)| \leq M$, for all $t \in [a, b]$, $a < b$, $a, b \in I$, then the following inequality holds*

$$\left| f(x) - \frac{1}{b-a} \int_a^b f(t) dt \right| \leq M(b-a) \left[\frac{1}{4} + \frac{\left(x - \frac{a+b}{2}\right)^2}{(b-a)^2} \right],$$

where $x \in [a, b]$.

[◇]Daniel Ianoși, Technical University of Cluj-Napoca
email: dan.ianosi@yahoo.com

This theorem has raised the interest of many researchers over time and has been generalized in various ways.

In [7], B.G. Pachpatte proved the following theorem which established a new Ostrowski type inequality involving functions whose modulus of the derivatives are convex.

Theorem 1.2 (see [7]) *Let $f, g : [a, b] \rightarrow \mathbb{R}$ be two absolutely continuous functions on $[a, b]$. If $|f'|, |g'|$ are convex function on $[a, b]$, then*

$$\begin{aligned} & \left| f(x)g(x) - \frac{1}{2(b-a)} \left[g(x) \int_a^b f(t)dt + f(x) \int_a^b g(t)dt \right] \right| \\ & \leq \frac{b-a}{4} \left[\frac{1}{4} + \frac{\left(x - \frac{a+b}{2}\right)^2}{(b-a)^2} \right] \\ & \quad \times [|g(x)| (f'(x) + \|f'\|) + |f(x)| (g'(x) + \|g'\|)], \quad x \in [a, b]. \end{aligned}$$

For $g(x) = 1$ in Theorem 1.2 is obtained the inequality proved in [1] (see Theorem 2 in [1]). This theorem is presented in the following.

Theorem 1.3 (see [1]) *Let $f : [a, b] \rightarrow \mathbb{R}$ be absolutely continuous function on $[a, b]$, $f' \in L_\infty[a, b]$. If $|f'|$ is convex function on (a, b) , then*

$$\begin{aligned} & \left| f(x) - \frac{1}{b-a} \int_a^b f(t)dt \right| \\ & \leq \frac{b-a}{2} \left[\frac{1}{4} + \frac{\left(x - \frac{a+b}{2}\right)^2}{(b-a)^2} \right] (\|f'(x)\| + \|f'\|_\infty). \end{aligned}$$

In 1978, W.W. Breckner (see [2]) introduced the concept of s -convex function as a generalization of convexity of functions.

Definition 1.1 *The function $f : [0, \infty) \rightarrow \mathbb{R}$ is called s -convex in the second sense if for any $t \in [0, 1]$, for any points $x, y \in [0, \infty)$ and $s \in (0, 1]$ fixed we have*

$$f(tx + (1-t)y) \leq t^s f(x) + (1-t)^s f(y).$$

Remark 1.1 For $s = 1$ in the above inequality it is obtained the convexity of function f in classic way:

$$f(tx + (1 - t)y) \leq tf(x) + (1 - t)f(y), \quad \forall t \in [0, 1], \quad x, y \in [0, \infty).$$

Definition 1.2 The function $f : I \rightarrow [0, \infty)$ is called p -function if for any points $x, y \in I$ and $t \in [0, 1]$ we have

$$f(tx + (1 - t)y) \leq f(x) + f(y).$$

Definition 1.3 (see [9]) Let I be an interval in $\mathbb{R}$. A function $f : I \rightarrow \mathbb{R}$ is called Godunova-Levin function if for any $x, y \in I$ and for any $t \in (0, 1)$ we have

$$f(tx + (1 - t)y) \leq \frac{f(x)}{t} + \frac{f(y)}{1 - t}.$$

In 2007, S. Varošanec (see [9]) introduced a large class of non-negative functions, called h -convex functions. This class contains several well-known classes of functions such as non-negative convex functions. This class is defined in the following way.

Definition 1.4 A non-negative function $f : I \rightarrow \mathbb{R}$ is called h -convex function, if for a function $h : J \rightarrow \mathbb{R}_+$, where J is an interval which contains $(0, 1)$, for any points $x, y \in I$ and $t \in (0, 1)$ we have

$$(1.1) \quad f(tx + (1 - t)y) \leq h(t)f(x) + h(1 - t)f(y).$$

Remark 1.2 From the above it is easy to see the following.

a) If $h(t) = t$, then the inequality (1.1) from Definition 1.4 gives the non-negative convex function in the classic way.

b) If $h(t) = \frac{1}{t}$, then the inequality from Definition 1.4 gives Godunova-Levin functions.

c) If $h(t) = 1$, then the inequality from Definition 1.4 gives p -functions.

d) If $h(t) = t^s, s \in (0, 1)$, then the inequality (1.1) from Definition 1.4 gives s -convex function in the second sense. (W. Breckner)

Note that h -convex functions have properties that classical convex functions do not have.

In [9], S. Varošanec gave some examples for h -convexity.

Example 1.1 Let $h_k : (0, \infty) \rightarrow \mathbb{R}$ be a non-negative function defined by

$$h_k(x) = x^k, \quad k \in \mathbb{R}.$$

For $k \leq 1$ we have $h_k(t) \geq t$, $\forall t \in (0, 1)$. Let $f : (0, \infty) \rightarrow \mathbb{R}$ be a non-negative function defined by $f(x) = x^q$, $q \in \mathbb{R}$, which is convex function for $q \in (-\infty, 0] \cup [1, \infty)$ and is concave function for $q \in (0, 1)$. For $t \in (0, 1)$ and any $x, y \in (0, \infty)$ we have the following inequalities

$$f(tx + (1-t)y) \leq tf(x) + (1-t)f(y) \leq h(t)f(x) + h(1-t)f(y).$$

Then the function f is h_k -convex if

- 1) $k \leq 1$ and $q \in (-\infty, 0] \cup [1, \infty)$ or
- 2) $k \leq q$ and $q \in (0, 1)$.

Example 1.2 Let $h_k : (0, \infty) \rightarrow \mathbb{R}$ be a non-negative function defined by

$$h_k(x) = x^k, \quad k < 0$$

and let $f : [a, b] \rightarrow \mathbb{R}$ be a function defined by

$$f(x) = \begin{cases} 1, & x \in [a, \frac{a+b}{2}) \cup (\frac{a+b}{2}, b] \\ 2^{1-k}, & x = \frac{a+b}{2}. \end{cases}$$

We note that the f is h_k -convex function, but it is not a continuous function.

The aim of this paper is to establish Ostrowski's inequality for fractional integrals and to give some applications based on linear functionals defined by the ψ -Riemann-Liouville fractional integrals and using Peano's theorem.

2 Ostrowski inequality using Peano's representation formula

In 1913, Giuseppe Peano demonstrated the following formula for the representation of linear functionals.

Theorem 2.1 (see [4], [8]) *Let $R : AC^m \rightarrow \mathbb{R}$ be a linear functional with the property $R(e_i) = 0$, $i = \overline{0, m-1}$, $m \in \mathbb{N}^*$. Then the following representation of the functional R is valid for any $f \in AC^m$*

$$(2.1) \quad R(f) = \frac{1}{(m-1)!} \int_a^b R(\varphi_t) f^{(m)} dt.$$

We used the function $\varphi_t : [a, b] \rightarrow \mathbb{R}$ defined by

$$\varphi_t(x) = (x-t)_+^{m-1} = \left(\frac{x-t+|x-t|}{2} \right)^{m-1}, \quad m \in \mathbb{N}^*.$$

For $m = 1$ it is obtained Heaviside's step function, defined by

$$\sigma(x-t) = (x-t)_+^0 = \begin{cases} 1, & \text{if } x \geq t \\ 0, & \text{if } x < t. \end{cases}$$

Using that $|f^{(m)}|$ is h -convex function in the relation (2.1) it is easy to prove the following theorem.

Theorem 2.2 *Let $R : AC^m \rightarrow \mathbb{R}$ be a linear functional with the property*

$$R(e_i) = 0, \quad i = \overline{0, m-1}, \quad m \in \mathbb{N}^*.$$

Then for any $f \in AC^m$ such that $|f^{(m)}|$ is h -convex function, the following inequality holds

$$(2.2) \quad |R(f)| \leq \frac{b-a}{(m-1)!} \int_0^1 \left| R_x \left(\left(\frac{x-a-u(b-a)+|x-a-u(b-a)|}{2} \right)^{m-1} \right) \right| \times \left(h(u) |f^{(m)}(b)| + h(1-u) |f^{(m)}(a)| \right) du.$$

Proof. Using the change of variable

$$\frac{t-a}{b-a} = u,$$

the equality (2.1) can be written as

$$R(f) = \frac{b-a}{(m-1)!} \int_a^b R_x(\varphi_{a+u(b-a)}) \cdot f^{(m)}(a+u(b-a)) du$$

or in an explicit form

$$R(f) = \frac{b-a}{(m-1)!} \times \int_a^b R_x \left(\left(\frac{x-a-u(b-a) + |x-a-u(b-a)|}{2} \right)^{m-1} \right) f^{(m)}(ub+(1-u)a) du.$$

From the above we obtain (2.2).

Remark 2.1 *The above inequality is equivalent with the following*

$$|R(f)| \leq \frac{1}{(m-1)!} \int_a^b |R_x(x-t)_+^{m-1}| \times \left(h \left(\frac{t-a}{b-a} \right) |f^{(m)}(b)| + h \left(\frac{b-t}{b-a} \right) |f^{(m)}(a)| \right) dt.$$

For $m = 1$, from the above we obtain the following inequality

$$(2.3) \quad |R(f)| \leq \int_a^b |R_x(\sigma(x-t))| \times \left(h \left(\frac{t-a}{b-a} \right) |f'(b)| + h \left(\frac{b-t}{b-a} \right) |f'(a)| \right) dt.$$

3 Applications of the Peano's theorem

Let R be a linear functional given by

$$R(f) = f(x) - \frac{1}{b-a} \int_a^b f(t) dt = \int_a^b R(\sigma(x-t)) f'(t) dt,$$

where

$$\begin{aligned} R(\sigma(\cdot - t)) &= \sigma(x - t) - \frac{1}{b - a} \int_a^b \sigma(u - t) du \\ &= \sigma(x - t) - \frac{1}{b - a} \int_a^b du = \sigma(x - t) - \frac{b - t}{b - a}. \end{aligned}$$

Using these, we have obtained the following theorem, which gives Ostrowski's inequality to functional

$$A(f) = f(x) - \frac{1}{b - a} \int_a^b f(t) dt.$$

From here it is obtained

$$|R(f)| = \left| \int_a^b \left(\sigma(x - t) - \frac{b - t}{b - a} \right) f'(t) dt \right|,$$

so

$$|R(f)| = (b - a) \left| \int_0^1 (\sigma(x - a - y(b - a) - 1 + y)) f'(yb + (1 - y)a) dy \right|,$$

where we used the change of variable $\frac{t-a}{b-a} = y$.

Using these, we have obtained the following theorem, which gives Ostrowski's inequality for the functional

$$A(f) = \frac{1}{b - a} \int_a^b f(t) dt.$$

Theorem 3.1 *Let $A : C[a, b] \rightarrow \mathbb{R}$ be a linear functional given by*

$$A(f) = f(x) - \frac{1}{b - a} \int_a^b f(t) dt.$$

Then for any f derivable on $[a, b]$, such that $|f'|$ is h -convex function, the following inequality holds

$$\begin{aligned} \left| f(x) - \frac{1}{b-a} \int_a^b f(t) dt \right| &\leq \int_a^b \left[\sigma(x-t) \frac{t-a}{b-a} + \sigma(t-x) \frac{b-t}{b-a} \right] \\ &\times \left[h\left(\frac{t-a}{b-a}\right) |f'(b)| + h\left(\frac{b-t}{b-a}\right) |f'(a)| \right] dt. \end{aligned}$$

Proof. From

$$\begin{aligned} R(f) &= f(x) - A(f) = f(x) - \frac{1}{b-a} \int_a^b f(t) dt \\ &= \int_a^b \left(\sigma(x-t) - \frac{b-t}{b-a} \right) f'(t) dt \end{aligned}$$

and using the change of variable $y = \frac{t-a}{b-a}$ and for $|f'|$ being h -convex function, we obtain

$$\begin{aligned} |R(f)| &= (b-a) \left| \int_0^1 [\sigma(x-a-y(b-a)) - 1+y] f'(yb+(1-y)a) dy \right| \\ &\leq (b-a) \int_0^1 |\sigma(x-a-y(b-a)) - 1+y| |f'(yb+(1-y)a)| dy \\ &\leq (b-a) \int_0^1 |\sigma(x-a-y(b-a)) - 1+y| [h(y) |f'(b)| + h(1-y) |f'(a)|] dy. \end{aligned}$$

For $t = a + y(b-a)$, we have

$$|R(f)| \leq \int_a^b \left| \sigma(x-t) - \frac{b-t}{b-a} \right| \left[h\left(\frac{t-a}{b-a}\right) |f'(b)| + h\left(\frac{b-t}{b-a}\right) |f'(a)| \right] dt$$

$$\begin{aligned}
&= \int_a^x \left| 1 - \frac{b-t}{b-a} \right| \left[h\left(\frac{t-a}{b-a}\right) |f'(b)| + h\left(\frac{b-t}{b-a}\right) |f'(a)| \right] dt \\
&\quad + \int_x^b \left| -\frac{t-b}{b-a} \right| \left[h\left(\frac{t-a}{b-a}\right) |f'(b)| + h\left(\frac{b-t}{b-a}\right) |f'(a)| \right] dt \\
&= \int_a^b \left[\sigma(x-t) \frac{t-a}{b-a} + \sigma(t-x) \frac{b-t}{b-a} \right] \\
&\quad \times \left[h\left(\frac{t-a}{b-a}\right) |f'(b)| + h\left(\frac{b-t}{b-a}\right) |f'(a)| \right] dt.
\end{aligned}$$

Remark 3.1 For certain choices of the function h and we suppose that the derivative $f' : (a, b) \rightarrow \mathbb{R}$ is bounded on (a, b) , with

$$\|f'\|_\infty = \sup_{t \in (a, b)} |f'(t)| < \infty,$$

we get the following Ostrowski type inequalities.

a) For $h(t) = t$, from Theorem 3.1 we obtain the Ostrowski type inequality (see [1], [7]) for convexity in the classic way

$$\left| f(x) - \frac{1}{b-a} \int_a^b f(t) dt \right| \leq \frac{(x-a)^2 + (b-x)^2}{4(b-a)} (|f'(x)| + \|f'\|_\infty)$$

and for $x = \frac{a+b}{2}$ we have

$$\left| f(x) - \frac{1}{b-a} \int_a^b f(t) dt \right| \leq \frac{b-a}{8} (|f'(x)| + \|f'\|_\infty).$$

b) For $h(t) = \frac{1}{t}$, from Theorem 3.1 we have

$$\left| f(x) - \frac{1}{b-a} \int_a^b f(t) dt \right| \leq (b-a) \ln \frac{(b-a)^2}{4(b-x)(x-a)} (|f'(x)| + \|f'\|_\infty)$$

and for $x = \frac{a+b}{2}$ we have

$$\left| f(x) - \frac{1}{b-a} \int_a^b f(t) dt \right| \leq \frac{b-a}{8} \ln 4 (|f'(x)| + \|f'\|_\infty).$$

Theorem 3.2 Let $R : C[a, b] \rightarrow \mathbb{R}$ be a linear functional, given by

$$R(f) = f(x) - \frac{1}{b-a} \int_a^b f(t) dt,$$

the constants $p, q > 1$ such that $\frac{1}{p} + \frac{1}{q} = 1$ and $h \in L_q[a, b]$. Then for any f derivable on $[a, b]$ such that $|f'|$ is h -convex function, the following inequality holds

$$(3.1) \quad |R(f)| \leq \frac{[(x-a)^{p+1} + (b-x)^{p+1}]^{\frac{1}{p}}}{(p+1)^{\frac{1}{p}}(b-a)} \\ \times \left(\int_a^b \left(h \left(\frac{t-a}{b-a} \right) |f'(b)| + h \left(\frac{b-t}{b-a} \right) |f'(a)| \right)^q dt \right)^{\frac{1}{q}}.$$

Proof. From Theorem 3.1 and Hölder's inequality we have

$$\left| f(x) - \frac{1}{b-a} \int_a^b f(t) dt \right| \leq \left(\int_a^b \left| \sigma(x-t) \frac{t-a}{b-a} + \sigma(t-x) \frac{b-t}{b-a} \right|^p dt \right)^{\frac{1}{p}} \\ \times \left(\int_a^b \left(h \left(\frac{t-a}{b-a} \right) |f'(b)| + h \left(\frac{b-t}{b-a} \right) |f'(a)| \right)^q dt \right)^{\frac{1}{q}},$$

where we used that $|f'|$ is h -convex function.

Using the following equalities

$$\int_a^b \left(\sigma(x-t) \frac{t-a}{b-a} + \sigma(t-x) \frac{b-t}{b-a} \right)^p dt \\ = \int_a^x \frac{(t-a)^p}{(b-a)^p} + \int_x^b \frac{(b-t)^p}{(b-a)^p} = \frac{(x-a)^{p+1}}{(p+1)(b-a)^p} + \frac{(b-x)^{p+1}}{(p+1)(b-a)^p}$$

we have obtained the conclusion.

Remark 3.2 We obtained the Ostrowski type inequality proved in [1] (see Theorem 3 in [1]).

Let $f : [a, b] \rightarrow \mathbb{R}$ be an absolutely continuous function on $[a, b]$. If $|f'|$ is convex on $[a, b]$, $f' \in L_q[a, b]$ and $p, q > 1$ such that $\frac{1}{p} + \frac{1}{q} = 1$, then

$$\begin{aligned} & \left| f(x) - \frac{1}{b-a} \int_a^b f(t) dt \right| \\ & \leq \frac{[(x-a)^{p+1} + (b-x)^{p+1}]^{\frac{1}{p}}}{2(p+1)^{\frac{1}{p}}} (b-a)^{\frac{1}{p}} \| |f'(x)| + |f'| \|_q. \end{aligned}$$

From the inequality (3.1), for $p \rightarrow \infty$ we get

$$\begin{aligned} & \left| f(x) - \frac{1}{b-a} \int_a^b f(t) dt \right| \\ & \leq \frac{1}{b-a} \max(x-a, b-x) \int_a^b \left(h\left(\frac{t-a}{b-a}\right) |f'(b)| + h\left(\frac{b-t}{b-a}\right) |f'(a)| \right) dt. \end{aligned}$$

From the equalities

$$\int_a^b h\left(\frac{t-a}{b-a}\right) dt = \int_a^b h\left(\frac{b-t}{b-a}\right) dt = (b-a) \int_0^1 h(t) dt,$$

we obtained the following.

Corollary 3.1 *Under the terms of the Theorem 3.2 we have*

$$\begin{aligned} (3.2) \quad & \left| f(x) - \frac{1}{b-a} \int_a^b f(t) dt \right| \\ & \leq \max(x-a, b-x) (|f'(a)| + |f'(b)|) \int_0^1 h(t) dt. \end{aligned}$$

Remark 3.3 *For certain choices of h , from (3.2), we obtain the following inequalities.*

1) If $h(t) = t$, then

$$\left| f(x) - \frac{1}{b-a} \int_a^b f(t) dt \right| \leq \frac{1}{2} \max(x-a, b-x) (|f'(a)| + |f'(b)|).$$

2) If $h(t) = 1$, then

$$\left| f(x) - \frac{1}{b-a} \int_a^b f(t) dt \right| \leq \max(x-a, b-x) (|f'(a)| + |f'(b)|).$$

3) If $f(t) = t^s$, $s \in (0, 1)$, then

$$\left| f(x) - \frac{1}{b-a} \int_a^b f(t) dt \right| \leq \frac{1}{s+1} \max(x-a, b-x) (|f'(a)| + |f'(b)|).$$

4 Ostrowski type inequalities through Riemann-Liouville fractional integrals

Definition 4.1 (see [5]) Let (a, b) be a finite or infinite interval in $\mathbb{R}$ and $\alpha > 0$. Let $\psi \in C^1[a, b]$ be an increasing and positive function. The left-sided and right sided ψ -Riemann-Liouville fractional integral of a function f on $[a, b]$ are defined by

$$I_{a+}^{\alpha;\psi} f(x) = \frac{1}{\Gamma(\alpha)} \int_a^x \psi'(t) (\psi(x) - \psi(t))^{\alpha-1} f(t) dt$$

$$(4.1) \quad I_{b-}^{\alpha;\psi} f(x) = \frac{1}{\Gamma(\alpha)} \int_x^b \psi'(t) (\psi(t) - \psi(x))^{\alpha-1} f(t) dt,$$

where $\Gamma(\cdot)$ is Gamma function.

Remark 4.1 For $0 < a < b$, $\rho > 0$ and $\psi(t) = \frac{t^\rho}{\rho}$ the functionals (4.1) become the functionals which were considered by Katugampola (see [3]) defined by the next equalities

$${}^\rho I_{a+}^{\alpha;\psi} f(x) = \frac{\rho^{1-\alpha}}{\Gamma(\alpha)} \int_a^x t^{\rho-1} (x^\rho - t^\rho)^{\alpha-1} f(t) dt$$

$${}^{\rho}I_{b-}^{\alpha;\psi}f(x) = \frac{\rho^{1-\alpha}}{\Gamma(\alpha)} \int_x^b t^{\rho-1} (t^{\rho} - x^{\rho})^{\alpha-1} f(t) dt.$$

Now, let us consider the functional $A : C[a, b] \rightarrow \mathbb{R}$ defined by

$$(4.2) \quad A(f) = \frac{\Gamma(\alpha+1)}{2(b-a)^{\alpha}} \\ \times \left[I_{\psi^{-1}(a)+}^{\alpha;\psi} (f \circ \psi)(\psi^{-1}(b)) + I_{\psi^{-1}(b)-}^{\alpha;\psi} (f \circ \psi)(\psi^{-1}(a)) \right].$$

The functional (4.2) can be written as

$$(4.3) \quad A(f) = \frac{\alpha}{2(b-a)^{\alpha}} \left[\int_{\psi^{-1}(a)}^{\psi^{-1}(b)} \psi'(t) (b - \psi(t))^{\alpha-1} f(\psi(t)) dt \right. \\ \left. + \int_{\psi^{-1}(a)}^{\psi^{-1}(b)} \psi'(t) (\psi(t) - a)^{\alpha-1} f(\psi(t)) dt \right].$$

We have that the A is a linear and positive functional and

$$A(e_0) = \frac{\alpha}{2(b-a)^{\alpha}} \\ \times \left[\int_{\psi^{-1}(a)}^{\psi^{-1}(b)} \psi'(t) (b - \psi(t))^{\alpha-1} dt + \int_{\psi^{-1}(a)}^{\psi^{-1}(b)} \psi'(t) (\psi(t) - a)^{\alpha-1} dt \right] = 1.$$

The following theorem is an Ostrowski type inequality relative to the functions whose derivatives are h -convex.

Theorem 4.1 *Let $f : [a, b] \rightarrow \mathbb{R}$ be a differentiable mapping such that $f' \in L_1[a, b]$. If $|f'|$ is h -convex function on (a, b) , then the following inequality holds*

$$\begin{aligned}
(4.4) \quad & |f(x) - A(f)| \\
& \leq \frac{1}{2(b-a)^\alpha} \int_a^b \left[h\left(\frac{u-a}{b-a}\right) |f'(b)| + h\left(\frac{b-u}{b-a}\right) |f'(a)| \right] \\
& \quad \times [\sigma(x-u)((b-a)^\alpha + (u-a)^\alpha - (b-u)^\alpha) \\
& \quad + \sigma(x-u)((b-a)^\alpha + (b-u)^\alpha - (u-a)^\alpha)] du,
\end{aligned}$$

where $x \in [a, b]$.

Proof. For x fixed, $x \in (a, b)$ we denote by R_x the following functional

$$R_x = f(x) - A(f).$$

We have

$$R_x(e_0) = e_0 - A(e_0) = 0.$$

So, we have the relation (2.3):

$$|R(f)| \leq \int_a^b |R_x(\sigma(x-u))| \left(h\left(\frac{u-a}{b-a}\right) |f'(b)| + h\left(\frac{b-u}{b-a}\right) |f'(a)| \right) du$$

and we will calculate $R_x(\sigma(x-u))$.

Let us compute $A_x(\sigma(x-u))$. From (4.3) we have

$$\begin{aligned}
A_x(\sigma(x-u)) &= \frac{\alpha}{2(b-a)^\alpha} \left[\int_{\psi^{-1}(a)}^{\psi^{-1}(b)} \psi'(t) (b-\psi(t))^{\alpha-1} \sigma(x-u) dt \right. \\
& \quad \left. + \int_{\psi^{-1}(a)}^{\psi^{-1}(b)} \psi'(t) (\psi(t)-a)^{\alpha-1} \sigma(x-u) dt \right] \\
&= \frac{\alpha}{2(b-a)^\alpha} \left[\int_{\psi^{-1}(a)}^{\psi^{-1}(u)} \psi'(t) (b-\psi(t))^{\alpha-1} dt + \int_{\psi^{-1}(u)}^{\psi^{-1}(b)} \psi'(t) (\psi(t)-a)^{\alpha-1} dt \right]
\end{aligned}$$

from where we obtain

$$A_x(\sigma(x-u)) = \frac{(b-a)^\alpha + (b-u)^\alpha - (u-a)^\alpha}{2(b-a)^\alpha}$$

where we used that

$$\sigma(x-u) = \sigma(\psi(t)-u) = \begin{cases} 0, & \text{if } \psi(t) < u \\ 1, & \text{if } \psi(t) \geq u \end{cases} = \begin{cases} 0, & \text{if } t < \psi^{-1}(u) \\ 1, & \text{if } t \geq \psi^{-1}(u) \end{cases}$$

The inequality (2.3) for the functional $R_x = f(x) - A(f)$ becomes

$$\begin{aligned} |R_x(f)| &= |f(x) - R_x(f)| \\ &\leq \int_a^b \left| \sigma(x-u) - \frac{(b-a)^\alpha + (b-u)^\alpha - (u-a)^\alpha}{2(b-a)^\alpha} \right| \\ &\quad \times \left(h\left(\frac{u-a}{b-a}\right) |f'(b)| + h\left(\frac{b-u}{b-a}\right) |f'(a)| \right) du \end{aligned}$$

or

$$\begin{aligned} |R_x(f)| &= |f(x) - R_x(f)| \\ &\leq \int_a^x \left| 1 - \frac{(b-a)^\alpha + (b-u)^\alpha - (u-a)^\alpha}{2(b-a)^\alpha} \right| \\ &\quad \times \left(h\left(\frac{u-a}{b-a}\right) |f'(b)| + h\left(\frac{b-u}{b-a}\right) |f'(a)| \right) du \\ &\quad + \int_x^b \left| -\frac{(b-a)^\alpha + (b-u)^\alpha - (u-a)^\alpha}{2(b-a)^\alpha} \right| \\ &\quad \times \left(h\left(\frac{u-a}{b-a}\right) |f'(b)| + h\left(\frac{b-u}{b-a}\right) |f'(a)| \right) du. \end{aligned}$$

From the above, we have

$$\begin{aligned}
|R_x(f)| &\leq \int_a^x \left| \frac{(b-a)^\alpha + (u-a)^\alpha - (b-u)^\alpha}{2(b-a)^\alpha} \right| \\
&\times \left(h\left(\frac{u-a}{b-a}\right) |f'(b)| + h\left(\frac{b-u}{b-a}\right) |f'(a)| \right) du \\
&+ \int_x^b \left| \frac{(b-a)^\alpha + (b-u)^\alpha - (u-a)^\alpha}{2(b-a)^\alpha} \right| \\
&\times \left(h\left(\frac{u-a}{b-a}\right) |f'(b)| + h\left(\frac{b-u}{b-a}\right) |f'(a)| \right) du \\
&\leq \int_a^b \left(h\left(\frac{u-a}{b-a}\right) |f'(b)| + h\left(\frac{b-u}{b-a}\right) |f'(a)| \right) \\
&\times \left[\sigma(x-u) \frac{(b-a)^\alpha + (u-a)^\alpha - (b-u)^\alpha}{2(b-a)^\alpha} \right. \\
&\quad \left. + \sigma(u-x) \frac{(b-a)^\alpha + (b-u)^\alpha - (u-a)^\alpha}{2(b-a)^\alpha} \right].
\end{aligned}$$

The proof is complete.

Remark 4.2 For $\alpha = 1$, the functional $A(f)$ is

$$A(f) = \frac{1}{b-a} \int_{\psi^{-1}(a)}^{\psi^{-1}(b)} \psi'(t) (f \circ \psi)(t) dt = \frac{1}{b-a} \int_a^b f(t) dt$$

and the inequality (4.4) is the Ostrowski inequality.

In the following we consider $\psi : [a, b] \rightarrow [0, 1]$ an increasing function, $\psi \in C^1[a, b]$ and we denote by F_ψ the following set

$$F_\psi = \{f : [\psi(a), \psi(b)] \rightarrow \mathbb{R}, f \in C^1[\psi(a), \psi(b)]\}.$$

Let us consider the linear functional $B : F_\psi \rightarrow \mathbb{R}$ defined by $B(f) = A(f(\psi))$, where A is a linear functional such that $A(e_0) = 1$.

Let $x \in (a, b)$ be a fixed point. The functional $R : F_\psi \rightarrow \mathbb{R}$ defined by

$$R(f) = f(\psi(x)) - B(f)$$

is a linear functional and $R(e_0) = 0$. By the Peano's theorem we have

$$(4.5) \quad \begin{aligned} R(f) &= \int_{\psi(a)}^{\psi(b)} R(\sigma(\cdot - u)) f'(u) du \\ &= \int_{\psi(a)}^{\psi(b)} [\sigma(\psi(x) - u) - A(\sigma(\psi(x) - u))] f'(u) du. \end{aligned}$$

If we put

$$\frac{u - \psi(a)}{\psi(b) - \psi(a)} = \psi(t),$$

then (4.5) becomes

$$\begin{aligned} R(f) &= (\psi(b) - \psi(a)) \int_0^1 [\sigma(\psi(x) - \psi(t)\psi(b) - (1 - \psi(t))\psi(a)) \\ &\quad - A(\sigma(\psi(x) - \psi(t)\psi(b) - (1 - \psi(t))\psi(a)))] \\ &\quad \times f'(\psi(t)\psi(b) + (1 - \psi(t))\psi(a)) dt. \end{aligned}$$

Let we suppose that $|f'|$ is h -convex function. Then from the above we obtain

$$\begin{aligned} |R(f)| &\leq (\psi(b) - \psi(a)) \int_0^1 |\sigma(\psi(x) - \psi(t)\psi(b) - (1 - \psi(t))\psi(a)) \\ &\quad - A(\sigma(\psi(x) - \psi(t)\psi(b) - (1 - \psi(t))\psi(a)))| \\ &\quad \times (|f'(\psi(b))| h(\psi(t)) + |f'(\psi(a))| h(1 - \psi(t))) \psi'(t) dt. \end{aligned}$$

So, we have

$$(4.6) \quad \begin{aligned} |R(f)| &\leq \int_{\psi(a)}^{\psi(b)} |\sigma(\psi(x) - u) - A(\sigma(\psi(x) - u))| \\ &\quad \times \left[|f'(\psi(a))| h\left(\frac{\psi(b) - u}{\psi(b) - \psi(a)}\right) + |f'(\psi(b))| h\left(\frac{u - \psi(a)}{\psi(b) - \psi(a)}\right) \right] du, \end{aligned}$$

where we used $\psi(a) + \psi(t)(\psi(b) - \psi(a)) = u$.

Let us consider the following linear positive functional:

$$B(f) = \frac{\Gamma(\alpha + 1)}{2} \left[\frac{I_{x^-}^{\alpha;\psi}(f \circ \psi)(a)}{(\psi(x) - \psi(a))^\alpha} + \frac{I_{x^+}^{\alpha;\psi}(f \circ \psi)(b)}{(\psi(b) - \psi(x))^\alpha} \right].$$

We remark that $B(e_0) = 1$ and so the functional $R(f) = f(\psi(x)) - B(f)$ has the degree of exactness 0. In the following we compute $I_{x^-}^{\alpha;\psi}(\sigma(\psi - u))(a)$ and $I_{x^+}^{\alpha;\psi}(\sigma(\psi - u))(b)$.

We have

$$\begin{aligned} I_{x^-}^{\alpha;\psi}(\sigma(\psi - u))(a) &= \frac{1}{\Gamma(\alpha)} \int_a^x \psi'(t) (\psi(t) - \psi(a))^{\alpha-1} \sigma(\psi(t) - u) dt \\ &= \frac{1}{\Gamma(\alpha)} \int_a^{\psi^{-1}(u)} \psi'(t) (\psi(t) - \psi(a))^{\alpha-1} \sigma(\psi(t) - u) dt \\ &\quad + \frac{1}{\Gamma(\alpha)} \int_{\psi^{-1}(u)}^x \psi'(t) (\psi(t) - \psi(a))^{\alpha-1} \sigma(\psi(t) - u) dt \\ &= \frac{1}{\Gamma(\alpha)} \int_{\psi^{-1}(u)}^x \psi'(t) (\psi(t) - \psi(a))^{\alpha-1} dt \\ &= \frac{1}{\Gamma(\alpha)} \frac{(\psi(x) - \psi(a))^\alpha - (u - \psi(a))^\alpha}{\alpha}, \end{aligned}$$

where

$$\sigma(\psi(t) - u) = \begin{cases} 0, & \text{if } t < \psi^{-1}(u) \\ 1, & \text{if } t \geq \psi^{-1}(u). \end{cases}$$

So, we have

$$(4.7) \quad I_{x^-}^{\alpha;\psi}(\sigma(\psi - u))(a) = \begin{cases} \frac{(\psi(x) - \psi(a))^\alpha - (u - \psi(a))^\alpha}{\Gamma(\alpha+1)}, & \text{if } u \leq \psi(x) \\ 0, & \text{if } u > \psi(x) \end{cases}$$

$$\begin{aligned}
I_{x^+}^{\alpha;\psi}(\sigma(\psi - u))(b) &= \frac{1}{\Gamma(\alpha)} \int_x^b \psi'(t) (\psi(b) - \psi(t))^{\alpha-1} \sigma(\psi(t) - u) dt \\
&= \frac{1}{\Gamma(\alpha)} \int_x^{\psi^{-1}(u)} \psi'(t) (\psi(b) - \psi(t))^{\alpha-1} \sigma(\psi(t) - u) dt \\
&\quad + \frac{1}{\Gamma(\alpha)} \int_{\psi^{-1}(u)}^b \psi'(t) (\psi(b) - \psi(t))^{\alpha-1} \sigma(\psi(t) - u) dt \\
&= \frac{1}{\Gamma(\alpha)} \int_{\psi^{-1}(u)}^b \psi'(t) (\psi(b) - \psi(t))^{\alpha-1} dt = \frac{(\psi(b) - u)^\alpha}{\Gamma(\alpha + 1)},
\end{aligned}$$

if $u > \psi(x)$.

If $u \leq \psi(x)$, then we have

$$\begin{aligned}
I_{x^+}^{\alpha;\psi}(\sigma(\psi - u))(b) &= \frac{1}{\Gamma(\alpha)} \int_x^b \psi'(t) (\psi(b) - \psi(t))^{\alpha-1} \sigma(\psi(t) - u) dt \\
&= \frac{(\psi(b) - \psi(x))^\alpha}{\Gamma(\alpha + 1)}.
\end{aligned}$$

So, we have

$$(4.8) \quad I_{x^+}^{\alpha;\psi}(\sigma(\psi - u))(b) = \begin{cases} \frac{(\psi(b) - \psi(x))^\alpha}{\Gamma(\alpha + 1)}, & \text{if } u \leq \psi(x) \\ \frac{(\psi(b) - u)^\alpha}{\Gamma(\alpha + 1)}, & \text{if } u > \psi(x) \end{cases}$$

Using (4.7) and (4.8), we obtain

$$\begin{aligned}
(4.9) \quad B(\sigma(\cdot - u)) &= \frac{\Gamma(\alpha + 1)}{2} \\
&\quad \times \left[\frac{I_{x^-}^{\alpha;\psi}(\sigma(\psi - u))(a)}{(\psi(x) - \psi(a))^\alpha} + \frac{I_{x^+}^{\alpha;\psi}(\sigma(\psi - u))(b)}{(\psi(b) - \psi(x))^\alpha} \right] \\
&= \frac{1}{2} \left[1 - \frac{(u - \psi(a))^\alpha}{(\psi(x) - \psi(a))^\alpha} + \sigma(u - \psi(x)) \frac{(\psi(b) - u)^\alpha}{(\psi(b) - \psi(x))^\alpha} + \sigma(\psi(x) - u) \right].
\end{aligned}$$

From the relations (4.6) and (4.9) we obtain the following inequality.

Theorem 4.2 *Let $f : [a, b] \rightarrow \mathbb{R}$ be a differentiable mapping such that $f' \in L_1[a, b]$. If $|f'|$ is h -convex function on (a, b) , then the following inequality holds*

$$\begin{aligned} & \left| f(\psi(x)) - \frac{\Gamma(\alpha + 1)}{2} \left[\frac{I_{x^-}^{\alpha; \psi}(f \circ \psi)(a)}{(\psi(x) - \psi(a))^\alpha} + \frac{I_{x^+}^{\alpha; \psi}(f \circ \psi)(b)}{(\psi(b) - \psi(x))^\alpha} \right] \right| \\ & \leq \frac{1}{2} \left\{ \int_{\psi(a)}^{\psi(x)} \frac{(u - \psi(a))^\alpha}{(\psi(x) - \psi(a))^\alpha} \left[h \left(\frac{\psi(b) - u}{\psi(b) - \psi(a)} \right) + h \left(\frac{u - \psi(a)}{\psi(b) - \psi(a)} \right) \right] du \right. \\ & \quad + \int_{\psi(x)}^{\psi(b)} \left(1 + \frac{(\psi(b) - u)^\alpha}{(\psi(b) - \psi(x))^\alpha} \right) \\ & \quad \times \left. \left[h \left(\frac{\psi(b) - u}{\psi(b) - \psi(a)} \right) + h \left(\frac{u - \psi(a)}{\psi(b) - \psi(a)} \right) \right] du \right\} \cdot \|f'\|_\infty. \end{aligned}$$

References

- [1] N.S. Barnett, P. Cerone, S.S. Dragomir, M.R. Pinheiro and A. Sofo. Ostrowski type inequalities for functions whose moduli of derivatives are convex, *RGMIA Res. Rep. Coll.*, **5**(2) (2002), 219-231.
- [2] W.W. Breckner. Stetigkeitsaussagen für eine Klasse verallgemeinerter konvexer funktionen in topologischen linearen Räumen, *Publ. Inst. Math.*, **23** (1978), 13-20.
- [3] G. Farid, U. Katugampola and M. Usman. Ostrowski-type fractional integral inequalities for mappings whose derivatives are h -convex via Katugampola fractional integrals, *Stud. Univ. Babeş-Bolyai Math.*, **63** (2018), no. 4, 465-474.
- [4] I. Gavrea and M. Ivan. A sharp estimate for the Peano error representation, *Applied Mathematics and Computation*, **252** (2015) 14-19.

-
- [5] K. Liu, J. Wang and D. O'Regan. On the Hermite-Hadamard type inequality for ψ -Riemann-Liouville fractional integrals via convex functions, *Journal of Inequalities and Applications*, (2019).
 - [6] A. Ostrowski. Über die Absolutabweichung einer differentierbaren funktion von ihrem integralmittelwert, *Comment. Math. Helv.*, **10** (1938), no. 1, 226-227.
 - [7] B.G. Pachpatte. On Ostrowski-Gruss-Cebysev type inequalities for functions whose moduli of derivatives are convex, *Journal of Inequalities in Pure and Applied Mathematics*, **6** (2005), no. 4, art. 128.
 - [8] G. Peano. Resto nelle formule di quadratura espresso con un integrale definito, *Atti della Accademia dei Lincei, Rendiconti*, **22** (1913), (ser. 5), 562-569.
 - [9] S. Varošanec. On h -convexity, *J. Math. Anal. Appl.*, 2007.