

The case of unbounded solutions set for unbounded feasible set in linear programming

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ABSTRACT. The case of the unbounded feasible set may lead to four possible exits. When a solution exists this might be unique, a bounded set not a singleton, or an unbounded set. The extended simplex algorithm, the present new variant looks for solving the linear programming problem. In particular, the algorithm is able to enumerate all the possible optimal solutions associated with the optimal basis, and returns the set of solutions.

KEY WORDS: Linear programming, pivoting algorithm, simplex algorithm, multiple solutions.

1 Introduction: the solutions set, examples

Although the topic may be considered obsolete, the literature enclosed in the list of references from [1] enumerates old and new titles and different approaches, the novelty here consists in returning the set of solutions $\mathcal{S}$, for the addressed problem. From the author's knowledge, there is no other

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article asserting that the linear programming problem is closed, in the sense that all solutions are found by an algorithm. Starting with a simple example, then detecting a direction, if exists, in the case of the unbounded admissible set, makes the motivation. The unique purpose of the present article is to give theoretical results that establish, along with the ones very-well known related to the existence, the uniqueness of solution, the boundedness, or the unboundedness criteria. Therefore, the standard linear simplex algorithm is enriched by the returns, consequences of the propositions the establishes the property of the optimal basis.

The structure of the present paper is mainly as expressed previously. In the present section three new characteristics of an optimal basis are used, called 0–max dual feasible, 0–min primal feasible, and 0–optimal. In the first result, the proposition states the criteria for the uniqueness or non-uniqueness of the solution. Then, in the second proposition, the property of the optimal basis may indicate when the set of solutions is unbounded, and the unboundedness direction is given, the vector denoted by $\bar{c}$. Although may look like a section of a textbook, the examples are given as they occur to obtain the concrete result, the general form of $\mathcal{S}$. In the second section, the cone of positive directions is given, denoted by $A_c^\perp$. The third section contain the main theorem, the extended simplex algorithm, and several examples that suggest nontriviality. The last section contains the form of $\mathcal{S}$, the utility for a system of linear inequalities, and possible extensions.

The standard form of a linear programming problem (LP) is

$$\min_{x \in S} \langle c, x \rangle,$$

where $S = \{x = (x_1, \dots, x_n) \in \mathbb{R}^n \mid A \cdot x^T = b^T, x \geq_{\mathbb{R}^n} 0_{\mathbb{R}^n}\}$, $n \geq 2$, with data $c \in \mathbb{R}^n$, $A = (A^j)_{1 \leq j \leq n} \in M_{m,n}(\mathbb{R})$, where A^j is the j –th column of A , and $b \in \mathbb{R}^m$ given. Denote by $\mathcal{S}$ the set of its solutions. The paper addresses a well studied topic, today present in exercises sections of standard textbooks (see for example [1], [4]), except that the complete solving is not contained. There should be no confusion between "unbounded optimal objective value" (as usually appears in textbooks), consequently the (LP) has no solution vs. the present approach that contains the case of unbounded set

of solutions, consequently the set of solutions is infinite and unbounded. The motivation starts from the clear difference between the two expressions *finding a solution to the problem* and *solving the problem*, especially when the feasible set S is not bounded. An immediate application is the best approximation, i.e. for $x^* \in \mathbb{R}^n$ given, find a solution to the problem

$$\begin{cases} \|x - x^*\| \rightarrow \min \\ x \in \mathcal{S}. \end{cases}$$

A matrix B consisting of m independent columns of A , $m < n$, is called *basic*. The remaining columns of A that are not in B are said to be outside the basis or nonbasic. The usual notations are used, $\mathcal{B} = \{j \in \{1, 2, \dots, n\} \mid \exists k, A^j = B^k\}$, $\bar{\mathcal{B}} = \{1, 2, \dots, n\} \setminus \mathcal{B}$, so $\{A^j\}_{1 \leq j \leq n} = \{\{A^j\}_{j \in \mathcal{B}}, \{A^i\}_{i \in \bar{\mathcal{B}}}\}$. The linear combination of A^i , $i \in \bar{\mathcal{B}}$, is given by $A^i = \sum_{j \in \mathcal{B}} \alpha_{ij} A^j$; also α_{0j} are the coordinates of b , $b^T = \sum_{j \in \mathcal{B}} \alpha_{0j} A^j$, and $\alpha_{i0} = \sum_{j \in \mathcal{B}} \alpha_{ij} c_j - c_i$, $\forall i \in \bar{\mathcal{B}}$. A basic matrix B is said to be *primal feasible* if $\min_{j \in \mathcal{B}} \alpha_{0j} \geq 0$, and *dual feasible* if $\max_{i \in \bar{\mathcal{B}}} \alpha_{i0} \leq 0$, respectively. If B is primal feasible and dual feasible then B is called *optimal*. About an optimal basis B , we say that it is *0-max dual feasible* if $\max_{i \in \bar{\mathcal{B}}} \alpha_{i0} = 0$ and *0-min primal feasible* if $\min_{j \in \mathcal{B}} \alpha_{0j} = 0$, respectively. More, we say that B is *0-optimal* if it is 0-max dual feasible and 0-min primal feasible.

The preoccupation to find multiple solutions is not new [9]. More recently we have the following statement.

Proposition 1.1 ([3]) *Let $x^0 = x^B$ be a solution for (LP) and let B be a 0-optimal basis.*

Denote by $\bar{\mathcal{B}}_0 = \{i \in \bar{\mathcal{B}} \mid \alpha_{i0} = 0\}$ and $\mathcal{B}_0 = \{j \in \mathcal{B} \mid \alpha_{0j} = 0\}$. The following implications apply:

1. *if $\alpha_{\bar{i}j_0} > 0$, $\forall \bar{i} \in \bar{\mathcal{B}}_0$, $\forall j_0 \in \mathcal{B}_0$, then $\mathcal{S} = \{x^0\}$;*
2. *if exist $\bar{i} \in \bar{\mathcal{B}}_0$, $k \in \mathcal{B} \setminus \mathcal{B}_0$ such that $\alpha_{\bar{i}k} > 0$, then there exists solution $x^1 \neq x^0$.*

Once the algorithm is on, the boundedness of S is not necessarily known (unless clearly indicated). Suppose that a solution is found. An uniqueness

criteria is given (see [3], Proposition 2.1). If it is not the case, i.e. $\mathcal{S}$ is not a singleton, can the algorithm be continued? There are some situations that can be detected in order to differentiate how many pivoting steps should be applied on the optimal table or to obtain an unboundedness direction.

Proposition 1.2 ([3], Proposition 2.3) *Suppose that $x^0 = x^B$ is a solution with B a 0-max dual feasible basis.*

If, for some $\bar{i} \in \bar{\mathcal{B}}_0$, $\alpha_{\bar{i}k} \leq 0, \forall k \in \mathcal{B}$, then there exists $\bar{c} > 0_{\mathbb{R}^n}$ such that

$$\{x^0 + \alpha \cdot \bar{c} \mid \alpha \geq 0\} \subseteq \mathcal{S}.$$

The unboundedness direction $\bar{c} = (\bar{c}_1, \dots, \bar{c}_n)$ is given by

$$\bar{c}_j = \begin{cases} -\alpha_{\bar{i}j}, & j \in \mathcal{B} \setminus \mathcal{B}_{\bar{i}0} \\ 1, & j = \bar{i} \\ 0, & \text{otherwise,} \end{cases} \quad \text{where } \mathcal{B}_{\bar{i}0} = \{k \in \mathcal{B} \mid \alpha_{\bar{i}k} = 0\}.$$

Denote by $\bar{\mathcal{B}}_{0-} = \{\bar{i} \in \bar{\mathcal{B}}_0 \mid \alpha_{\bar{i}k} \leq 0, \forall k \in \mathcal{B}\}$. The previous statement asserts for a 0-max dual feasible basis, a sufficient condition for $\mathcal{S}$ to be unbounded is that $\bar{\mathcal{B}}_{0-} \neq \emptyset$.

$$\text{For } \begin{cases} -x_1 + x_2 \longrightarrow \min \\ x_1 - x_2 + x_3 = 0 \\ x_1, x_2, x_3 \geq 0, \end{cases} \quad \text{one has } \mathcal{B} = \{1\}, x^0 = (0, 0, 0), \bar{\mathcal{B}}_0 = \{2\},$$

for $\bar{i} \in \bar{\mathcal{B}}_0$, such that $\forall j \in \mathcal{B}, \alpha_{\bar{i}j} \leq 0, \mathcal{B}_{\bar{i}0} = \emptyset, \alpha_{2j} = -1$ (see Figure

$$1); \bar{c} = (\bar{c}_1, \bar{c}_2, \bar{c}_3) \text{ is given by } \bar{c}_j = \begin{cases} 1, & j = 1 \\ 1, & j = 2 \\ 0, & j = 3, \end{cases} \text{ thus } \bar{c} = (1, 1, 0). \text{ Since}$$

$x^1 = (1, 1, 0)$ and x^0 are comparable solutions in $\mathbb{R}^3$, $\bar{c}$ can also be taken $x^1 - x^0$. The set of solutions is

$$\begin{aligned} \mathcal{S} &= \{(\alpha, \alpha, 0) \mid \alpha \geq 0\} \\ &= \{(0, 0, 0) + \alpha \cdot (1, 1, 0) \mid \alpha \geq 0\} = \{x^0 + \alpha \cdot \bar{c} \mid \alpha \geq 0\}. \end{aligned}$$

Note that the conclusions on the structure of $\mathcal{S}$ depends on the nature of the optimal basis (see Theorem 3.1 below). More, if $\mathcal{S}$ is established to be unbounded, its algebraic dimension remains to be determined.

	A^1	
A^3	1	-1
A^2	-1	0
	0	

Figure 1: Optimal table; $x^0 = (0, 0, 0)$.

Let us consider the following example similar to the one above.

Example 1.1 *For the linear programming problem*

$$\begin{cases} -x_1 + x_2 - x_3 + x_4 \longrightarrow \min \\ x_1 - x_2 + x_3 - x_4 + x_5 = 0 \\ x_1, x_2, x_3, x_4, x_5 \geq 0, \end{cases}$$

by observation, it can be deduced that $\mathcal{S}$ contains

$$\{(\alpha_1, \alpha_1, \alpha_2, \alpha_2, 0) \mid \alpha_1, \alpha_2 \geq 0\}.$$

By computations, $\mathcal{B} = \{1\}$, $x^0 = (0, 0, 0, 0, 0)$, $\bar{\mathcal{B}}_0 = \{2, 3, 4\}$, for $\bar{i} \in \bar{\mathcal{B}}_0$, $\mathcal{B}_{i0} = \emptyset$ (see Figure 2). Nevertheless, from the optimal table, two

directions of unboundedness can be combined $\bar{c}_j^1 = \begin{cases} 1, & j = 1 \\ 1, & j = \bar{i} = 2 \\ 0, & j \in \{3, 4, 5\} \end{cases}$ thus

$$\bar{c}^1 = (1, 1, 0, 0, 0) \text{ and } \bar{c}_j^2 = \begin{cases} 1, & j = 1 \\ 1, & j = \bar{i} = 4 \\ 0, & j \in \{2, 3, 5\} \end{cases} \text{ thus } \bar{c}^2 = (1, 0, 0, 1, 0),$$

therefore we have

$$\begin{aligned} \mathcal{S} &= \{(x_1, x_2, x_3, x_4, 0) \in \mathbb{R}^5 \mid x_1 - x_2 + x_3 - x_4 = 0\} \\ &\supset \{(0, 0, 0, 0, 0) + \alpha_1 \cdot (1, 1, 0, 0, 0) + \alpha_2 \cdot (1, 0, 0, 1, 0) \mid \alpha_1, \alpha_2 \geq 0\}. \end{aligned}$$

By a pivoting operation, with $\alpha_{31} = 1$ as pivoting element, one gets

	A ¹			A ³	
A ⁵	1	-1	A ⁵	1	-1
A ²	-1	0	A ²	-1	0
A ³	1	0	A ¹	1	0
A ⁴	-1	0	A ⁴	-1	0
	0			0	

Figure 2: Optimal table; $x^0 = (0, 0, 0, 0, 0)$; $\bar{c}^1, \bar{c}^2; \bar{c}^3, \bar{c}^4$.

$\bar{c}^3 = (0, 1, 1, 0, 0)$ and $\bar{c}^4 = (0, 0, 1, 1, 0)$. Since $\bar{c}^1, \bar{c}^2$, and $\bar{c}^3$, for example, are linearly independent ($\bar{c}^1, \bar{c}^2, \bar{c}^3$, and $\bar{c}^4$ are not), one obtains

$$\mathcal{S} = x^0 + C,$$

where

$$C = \left\{ \sum_{i=1}^3 \alpha_i \cdot \bar{c}^i \mid \alpha_i \geq 0 \right\}$$

is the cone generated by $\{\bar{c}^i\}_{i \in \{1,2,3\}}$.

In the present article we try to answer to the following question: once an optimal table is obtained, with the associated optimal basis, how many solutions one can get out of it? Can $\mathcal{S}$ be completely determined? By intuition, yes, let say $\mathcal{S} = \text{co}\{x^k\}_{k \in K} + C$, where K, x^k , and C to be specified (by convention $\emptyset + \{0_{\mathbb{R}}\} = \emptyset$). Parts of the proofs are given in [3]. The answer is related to the algebraic dimension for the unboundedness if C is nonempty. At least a direction is provided for this, it remains that all to be found.

2 The cone of positive directions

Inspired by the Lemma 3.1 in [2], let us denote by

$$A_c^\perp = \{\bar{c} \in \mathbb{R}^n \mid \bar{c} \succ_{\mathbb{R}^n} 0_{\mathbb{R}^n} \text{ such that } \langle c, \bar{c} \rangle = 0, A \cdot \bar{c}^T = 0_{\mathbb{R}^m}^T\} \cup \{0_{\mathbb{R}^n}\}.$$

If $\bar{c}^1, \bar{c}^2 \in A_c^\perp$ then $\alpha_1 \cdot \bar{c}^1 + \alpha_2 \cdot \bar{c}^2 \in A_c^\perp$, for all $\alpha_1, \alpha_2 \geq 0$. Define $\dim A_c^\perp$ as the algebraic dimension of the vectorial subspace generated by $A_c^\perp$, i.e. $\dim A_c^\perp = \dim \langle A_c^\perp \rangle$. Example 1.1 considers $\dim A_c^\perp = 3$.

Example 2.1 Let $b_1, b_2 > 0$ and consider the following linear programming problem:

$$(2.1) \quad \begin{cases} x_1 + x_2 & \longrightarrow \min \\ x_1 & + x_3 - x_4 = b_1 \\ x_2 - x_3 + x_4 = b_2 \\ x_1, x_2, x_3, x_4 \geq 0. \end{cases}$$

Data are $c = (1, 1, 0, 0)$, $A = \begin{pmatrix} 1 & 0 & 1 & -1 \\ 0 & 1 & -1 & 1 \end{pmatrix} = (A^j)_{1 \leq j \leq 4}$, and $b^T = \begin{pmatrix} b_1 \\ b_2 \end{pmatrix}$. After the computational steps $B = \{A^1, A^2\}$ is optimal with solution $x^0 = (b_1, b_2, 0, 0)$ and $\bar{B}_0 = \{3, 4\}$. By a pivoting operation $\alpha_{31} = 1$, one gets the solution $x^1 = (0, b_1 + b_2, b_1, 0)$ and $\bar{B}_{0-} \neq \emptyset$. Hypotheses of Proposition 1.2 are satisfied, $\bar{i} = 4 \in \bar{B}_{0-}$ (see Figure 3). By the pivoting element $\alpha_{42} = 1$, one obtains the solution $x^2 = (b_1 + b_2, 0, 0, b_2)$. S contains $\text{co}\{x^0, x^1, x^2\}$.

1	A ¹	A ²	
A ³	1	-1	0
A ⁴	-1	1	0
	b ₁	b ₂	

2	A ³	A ²	
A ¹	1	1	0
A ⁴	-1	0	0
	b ₁	b ₁ +b ₂	

2'	A ¹	A ⁴	
A ³	0	-1	0
A ²	1	1	0
	b ₁ +b ₂	b ₂	

Figure 3: $a_3\beta, x^0, x^1, x^2, \bar{c}$

Solutions x^0 and x^1 are not comparable in $\mathbb{R}^4$, so $\bar{c}$ cannot be taken $x^0 - x^1$. By table 2, one has $\mathcal{B} = \{2, 3\}$, $\bar{i} = 4$, and $\mathcal{B}_{40} = \{2\}$, therefore $\bar{c} =$

$$(0, 0, 1, 1). \text{ Problem (2.1) reduces to } \begin{cases} x_3 - x_4 \leq b_1 \\ -x_3 + x_4 \leq b_2 \\ x_3, x_4 \geq 0. \end{cases} \quad \text{The set of solutions}$$

contains

$$\{x^1 + \alpha \cdot \bar{c} \mid \alpha \geq 0\} = \{(0, b_1 + b_2, b_1 + \alpha, \alpha) \mid \alpha \geq 0\}$$

and

$$\{x^2 + \alpha \cdot \bar{c} \mid \alpha \geq 0\} = \{(b_1 + b_2, 0, \alpha, b_2 + \alpha) \mid \alpha \geq 0\},$$

and it is

$$\text{co}\{x^0, x^1, x^2\} + C = \{(b_1 - \alpha + \beta, b_2 + \alpha - \beta, \alpha, \beta) \mid \alpha, \beta \geq 0, \alpha - \beta \in [-b_2, b_1]\},$$

where $C = A_c^\perp = \{\alpha \cdot \bar{c} \mid \alpha \geq 0\}$.

3 Extended simplex algorithm

Based on the result from [3] and the form of $\mathcal{S}$ we give the extended simplex algorithm that is supposed to find all solutions, consequently to determine $\mathcal{S}$. Let us remind the mentioned result.

Theorem 3.1 ([3], Theorem 4.4) *Suppose that B is the optimal basis. The following implications apply:*

1. *If B is NOT 0-max dual feasible then $\text{card } \mathcal{S} = 1$.*
2. *If B is 0-optimal basis and $\alpha_{\bar{i}j} > 0, \forall \bar{i} \in \bar{\mathcal{B}}_0, \forall j \in \mathcal{B}_0$, then $\text{card } \mathcal{S} = 1$.*
3. *If B is 0-optimal basis and exist $\bar{i} \in \bar{\mathcal{B}}_0, k \in \mathcal{B} \setminus \mathcal{B}_0$ such that $\alpha_{\bar{i}k} > 0$, then $\text{card } \mathcal{S} > 1$, with $\mathcal{S}$ bounded.*
4. *If B is 0-max dual feasible and $\bar{\mathcal{B}}_{0-} \neq \emptyset$, then $\mathcal{S}$ is unbounded.*

Simplex Algorithm 3.0

Step 1. Chose primal admissible basis B from $(A^j)_{1 \leq j \leq n} = A$. Determine α_{0j} ($j \in \mathcal{B}$) such that $b^T = \sum_{j \in \mathcal{B}} \alpha_{0j} A^j$;

Put $s := 0$;

Step 2. Determine α_{ij} ($i \in \bar{\mathcal{B}}, j \in \mathcal{B}$) such that $A^i = \sum_{j \in \mathcal{B}} \alpha_{ij} A^j$;

Step 3. Compute $\alpha_{i0} = \sum_{j \in \mathcal{B}} \alpha_{ij} c_j - c_i, \forall i \in \bar{\mathcal{B}}; \alpha_{00} = \sum_{j \in \mathcal{B}} \alpha_{0j} c_j$;

Step 4. Look for the signs of the numbers α_{i0} ($i \in \bar{\mathcal{B}}$)

- a) $\forall i \in \bar{\mathcal{B}}, \alpha_{i0} \leq 0$ then declare: " $x^s \in \mathbb{R}^n, x_j^s = \begin{cases} \alpha_{0j}, & j \in \mathcal{B} \\ 0, & j \in \bar{\mathcal{B}} \end{cases}$ is solution for (LP) " and α_{00} the minimum value of the objective function;
- a1) if $\max_{i \in \bar{\mathcal{B}}} \alpha_{i0} < 0$, then declare: " $\mathcal{S} = \{x^0\}$ ", STOP.
- a2) if $\max_{i \in \bar{\mathcal{B}}} \alpha_{i0} = 0 = \min_{j \in \mathcal{B}} \alpha_{0j}$ and $\alpha_{\bar{i}j} > 0, \forall \bar{i} \in \bar{\mathcal{B}}_0, \forall j \in \mathcal{B}_0$, then declare: " $\mathcal{S} = \{x^0\}$ ", STOP.
- a3) if $\max_{i \in \bar{\mathcal{B}}} \alpha_{i0} = 0$ and $\min_{j \in \mathcal{B}} \alpha_{0j} > 0$, then form the set

$$\bar{\mathcal{B}}_0 = \{i \in \bar{\mathcal{B}} \mid \alpha_{i0} = 0\} = \{\bar{i}_1, \bar{i}_2, \dots, \bar{i}_p\};$$

- a3 α) $\forall i \in \bar{\mathcal{B}}_0, \exists j \in \mathcal{B}$ such that $\alpha_{ij} > 0$; Chose an index $h \in \bar{\mathcal{B}}_{0+} = \{\bar{i} \in \bar{\mathcal{B}}_0 \mid \exists k \in \mathcal{B} \text{ with } \alpha_{\bar{i}k} > 0\}$;
 $s := s + 1$;
 If $s \leq p$ declare: "each element of $\text{co}\{x^0, x^s\}$ is solution for (LP) ", go to Step 6;
- a3 β) $\exists \bar{i} \in \bar{\mathcal{B}}_0$ such that $\forall j \in \mathcal{B}, \alpha_{\bar{i}j} \leq 0$. Define the set of indexes $\mathcal{B}_{\bar{i}0} = \{k \in \mathcal{B} \mid \alpha_{\bar{i}k} = 0\}$ and vector $\bar{c}^{\bar{i}} = (\bar{c}_1, \dots, \bar{c}_n)$ with

$$\bar{c}_j = \begin{cases} -\alpha_{\bar{i}j}, & j \in \mathcal{B} \setminus \mathcal{B}_{\bar{i}0} \\ 1, & j = \bar{i} \\ 0, & \text{otherwise.} \end{cases}$$

Form the set $\bar{\mathcal{B}}_{0-} = \{\bar{i} \in \bar{\mathcal{B}}_0 \mid \alpha_{\bar{i}k} \leq 0, \forall k \in \mathcal{B}\}$ and declare: " $\mathcal{S}$ is not bounded; $\mathcal{S} \supset x^0 + A_c^\perp$."

- b) $\exists i \in \bar{\mathcal{B}}, \alpha_{i0} > 0$ form the set

$$\bar{\mathcal{B}}_+ = \{i \in \bar{\mathcal{B}} \mid \alpha_{i0} > 0\};$$

Step 5. $\forall i \in \bar{\mathcal{B}}_+$, Look for the signs of the numbers α_{ij} ($j \in \mathcal{B}$)

- a) $\exists i \in \bar{\mathcal{B}}_+$ such that $\forall j \in \mathcal{B}, \alpha_{ij} \leq 0$ then declare: " $\mathcal{S} = \emptyset$."
- b) $\forall i \in \bar{\mathcal{B}}_+, \exists j \in \mathcal{B}$ such that $\alpha_{ij} > 0$, chose an index $h \in \bar{\mathcal{B}}_+$;

Step 6. Chose an index $k \in \mathcal{B}$ such that $\alpha_{hk} > 0$ and

$$\frac{\alpha_{0k}}{\alpha_{hk}} = \min \left\{ \frac{\alpha_{0j}}{\alpha_{hj}} \mid j \in \mathcal{B}, \alpha_{hj} > 0 \right\};$$

Step 7. Replace in B , vector A^k with A^h and go to Step 2.

Of course, any pivoting algorithm ([3, 8]) can conclude to these exits, precisely when $\mathcal{S}$ is a singleton, a bounded convex set (not a singleton) or an unbounded set.

By using the simplex algorithm 3.0 we give the following examples.

Example 3.1 *For the linear programming problem*

$$\begin{cases} -x_1 + 2x_2 + x_3 \longrightarrow \min \\ x_1 - 2x_2 - x_3 + x_4 = 0 \\ x_1, x_2, x_3, x_4 \geq 0, \end{cases}$$

$\mathcal{B} = \{1\}$, $x^0 = (0, 0, 0, 0)$, *there exists $\bar{i} = 2 \in \bar{\mathcal{B}}_0$ such that $\forall j \in \mathcal{B}, \alpha_{2j} \leq 0$;*
 $\bar{c}^1 = (1, 0, 1, 0)$, *similarly, for $\bar{i} = 3 \in \bar{\mathcal{B}}_{0-}$, $\bar{c}^2 = (2, 1, 0, 0)$, consequently*

	A^1	
A^4	1	-1
A^2	-2	0
A^3	-1	0
	0	

	A^1	
A^4	2	0
A^2	-2	0
A^3	-1	0
	2	

	A^4	
A^1	1/2	0
A^2	-1	0
A^3	-1/2	0
	1	

Figure 4: $a3\beta$; Example 3.1 $\bar{c}^1, \bar{c}^2$; Example 3.2 $\bar{c}^1, \bar{c}^2, \bar{c}^3, \bar{c}^4$

$$\mathcal{S} = x^0 + \left\{ \sum_{i=1}^2 \alpha_i \cdot \bar{c}^i \mid \alpha_i \geq 0 \right\} = \{(\alpha_1 + 2\alpha_2, \alpha_2, \alpha_1, 0) \mid \alpha_1, \alpha_2 \geq 0\}.$$

Example 3.2 *Let the linear programming problem*

$$\begin{cases} -x_1 + 2x_2 + x_3 - 2x_4 \longrightarrow \min \\ x_1 - 2x_2 - x_3 + 2x_4 = 2 \\ x_1, x_2, x_3, x_4 \geq 0, \end{cases}$$

$\mathcal{B} = \{1\}$, $x^0 = (2, 0, 0, 0)$, $\bar{\mathcal{B}}_0 = \{2, 3, 4\}$, there exists $\bar{i} = 2 \in \bar{\mathcal{B}}_0$ such that $\forall j \in \mathcal{B}$, $\alpha_{2j} \leq 0$; $\bar{c}^1 = (2, 1, 0, 0)$, and for $\bar{i} = 3$, $\bar{c}^2 = (1, 0, 1, 0)$, therefore $A_c^\perp \supset \{\bar{c}^1, \bar{c}^2\}$ (see Figure 4). By the pivoting element $\alpha_{41} = 2$ we get $x^1 = (0, 0, 0, 1)$ and $A_c^\perp \supset \{\bar{c}^3, \bar{c}^4\}$, with $\bar{c}^3 = (0, 1, 0, 1)$, $\bar{c}^4 = (0, 0, 1, 1/2)$. Consequently

$$\mathcal{S} = \text{co}\{x^0, x^1\} + \left\{ \sum_{i=1}^3 \alpha_i \cdot \bar{c}^i \mid \alpha_i \geq 0 \right\}.$$

Example 3.3 *Similar to Example 2.1, let $b_1, b_2, b_3 > 0$ and consider the following linear programming problem:*

$$\begin{cases} x_1 + x_2 + x_3 & \longrightarrow \min \\ x_1 & + x_4 - x_5 = b_1 \\ & x_2 & + x_5 - x_6 = b_2 \\ & & x_3 - x_4 & + x_6 = b_3 \\ x_1, x_2, x_3, x_4, x_5, x_6 \geq 0. \end{cases}$$

$B = \{A^1, A^2, A^3\}$ is an optimal basis with solution $x^0 = (b_1, b_2, b_3, 0, 0, 0)$ and $\bar{\mathcal{B}}_0 = \{4, 5, 6\}$. By successive pivoting operations, one gets solutions $x^1 = (0, b_2, b_1 + b_3, b_1, 0, 0)$, $x^2 = (0, 0, \sum b_i, b_1 + b_2, b_2, 0)$, $x^3 = (b_1 + b_2, 0, b_3, 0, b_2, 0)$, $x^4 = (0, \sum b_i, 0, b_1, 0, b_1 + b_3)$, $x^5 = (b_1, b_2 + b_3, 0, 0, 0, b_3)$, $x^6 = (\sum b_i, 0, 0, 0, b_2 + b_3, b_3)$; case $a3\beta$ occurs with $\bar{c} = (0, 0, 0, 1, 1, 1)$. One obtains

$$\mathcal{S} = \text{co}\{x^0, x^1, x^2, x^3, x^4, x^5, x^6\} + \{\alpha \cdot \bar{c} \mid \alpha \geq 0\}.$$

In general, for $b_1, \dots, b_n > 0$, the following linear programming problem

$$\begin{cases} \sum_{i=1}^n x_i & \longrightarrow \min \\ x_i & + x_{n+i} - x_{n+i+1} = b_i, \quad i \in \{1, \dots, n-1\} \\ x_n - x_{n+1} & + x_{2n} = b_n \\ x_1, \dots, x_{2n} & \geq 0, \end{cases}$$

has

$$\mathcal{S} = \text{co}\{x^k\}_{k \in K} + \{\alpha \cdot \bar{c} \mid \alpha \geq 0\},$$

with $\text{card } K = 2^n - 1$, $\bar{c} = \sum_{i=n+1}^{2n} e_i$, $e_i = (0, 0, \dots, 0, 1, 0, \dots, 0) \in \mathbb{R}^{2n}$.

Example 3.4 Let $p \in \mathbb{N}$, $p \geq 3$, $i_0 \in \{1, \dots, p\}$, $0 \leq c_j < c_{i_0} \leq c_1$, $\forall j \in \{1, \dots, p\} \setminus \{1, i_0\}$. Let the linear programming problem

$$\begin{cases} -(c_1 x_1 + \dots + c_p x_p) & \longrightarrow \min \\ x_1 + \dots + x_p + x_{p+1} & = 1 \\ x_1, \dots, x_p, x_{p+1} & \geq 0. \end{cases}$$

By the analytical approach, since $-(c_1 x_1 + \dots + c_p x_p) = -c_1 x_1 - (c_2 - c_1)x_2 - c_1 x_2 + \dots - (c_p - c_1)x_p - c_1 x_p$ attains its minimum if $\sum_{i=1}^p x_i = 1$, $x_i = 0$, $i \in \{2, \dots, p\}$, so $x_1 = 1$ and $x_{p+1} = 0$, the set of solutions can be provided. Alternatively, by computations (see Figure 5) $\mathcal{B} = \{1\}$, $x^0 = (1, 0, \dots, 0, 0) =$

$$e_1 \in \mathbb{R}^{p+1} \text{ is solution and } \bar{\mathcal{B}}_0 = \begin{cases} \{i_0\}, & c_{i_0} = c_1 \\ \emptyset, & c_{i_0} \neq c_1. \end{cases}$$

If $c_{i_0} = c_1$, take $\alpha_{i_0 1} = 1$ as pivoting element, so $x^{i_0} = (0, \dots, 0, 1, 0, \dots, 0) = e_{i_0} \in \mathbb{R}^{p+1}$ is solution. Then $\mathcal{S} = \begin{cases} \text{co}\{e_1, e_{i_0}\}, & c_{i_0} = c_1 \\ \{e_1\}, & c_{i_0} < c_1. \end{cases}$

Example 3.5 Let $p \in \mathbb{N}$, $p \geq 2$. Let the linear programming problem

$$\begin{cases} -\sum_{i=1}^p (x_{2i-1} - x_{2i}) & \longrightarrow \min \\ \sum_{i=1}^p (x_{2i-1} - x_{2i}) + x_{2p+1} & = 0 \\ x_1, \dots, x_{2p}, x_{2p+1} & \geq 0. \end{cases}$$

	A^1			A^i	
A^2	1	$c_2 - c_1$	A^2	1	$c_2 - c_{i0}$
...		...	...		...
A^{i0}	1	$c_{i0} - c_1$	A^1	1	$c_{i0} - c_1$
...	...	...	...	...	...
A^p	1	$c_p - c_1$	A^p	1	$c_p - c_{i0}$
A^{p+1}	1	$c_{p+1} - c_1$	A^{p+1}	1	$c_{p+1} - c_{i0}$
	1			1	

Figure 5: e_1, e_{i0}

For $p = 2$, Example 1.1 can be reproduced. By computations (see Figure 6), $x^0 = 0_{\mathbb{R}^{2p+1}}$, $\bar{B}_{0-} = \{2i \mid i \in \{1, \dots, p\}\}$, $\bar{c}^1 = (1, 1, 0, 0, \dots, 0, 0) \in \mathbb{R}^{2p+1}$, $\bar{c}^2 = (1, 0, 0, 1, \dots, 0, 0)$, ..., $\bar{c}^p = (1, 0, 0, 0, \dots, 1, 0) \in \mathbb{R}^{2p+1}$, $\bar{B}_{0+} = \{2i+1 \mid i \in \{1, \dots, p-1\}\}$. By successive pivoting operations from the first table, with $\alpha_{2k+1,1}$ as pivoting element, $k \in \{2, \dots, p-1\}$, we get $\tilde{c}^k = e_2 + e_{2k+1}$, $k \in \{1, \dots, p-1\}$. We prove that $\dim A_c^\perp = 2p-1$ (directly, one restriction gives the dimension of the hyperplane $k-1$, with $k=2p$ each variable doubled independently). Indeed, let λ_{2k} , $k \in \{1, \dots, p\}$, λ_{2k+1} , $k \in \{1, \dots, p-1\}$, be such that

$$\sum_{k=1}^p \lambda_{2k} \cdot (e_1 + e_{2k}) + \sum_{k=1}^{p-1} \lambda_{2k+1} \cdot (e_2 + e_{2k+1}) = 0_{\mathbb{R}^{2p+1}}.$$

It follows that $\lambda_{2k} = 0$, $\lambda_{2k+1} = 0$, $2 \leq k \leq p-1$, $\lambda_{2p} = 0$, $\lambda_3 = 0$. Also $\sum_{k=1}^p \lambda_{2k} = 0$, thus $\lambda_2 = 0$. Along with

$$\text{rank} \begin{pmatrix} e_1 + e_{2k} \\ e_1 + e_{2p} \\ e_2 + e_{2k+1} \end{pmatrix}_{k \in \{1, \dots, p-1\}} = 2p-1,$$

since

$$\begin{vmatrix} 1 & 1 & 0 & 0 & 0 & 0 & \dots & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 & 0 & 0 & \dots & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 1 & \dots & 0 & 0 & 0 \\ \dots & & & & & & & & & \\ 1 & 0 & 0 & 0 & 0 & 0 & \dots & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 & 0 & 0 & \dots & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 & \dots & 0 & 0 & 0 \\ \dots & & & & & & & & & \\ 0 & 1 & 0 & 0 & 0 & 0 & \dots & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & \dots & 0 & 0 & 1 \end{vmatrix} = 1 \neq 0,$$

we obtain that $\dim A_c^\perp = 2p - 1$.

	A^1			A^3			A^5			A^{2p-1}	
A^2	-1	0	A^2	-1	0	A^2	-1	0	A^2	-1	0
A^3	1	0	A^1	1	0	A^3	1	0	A^3	1	0
...	...	0	...	...	0	A^4	-1	0	A^4	-1	0
A^{2p}	-1	0	A^{2p}	-1	0	A^1	1	0	...	...	0
A^{2p+1}	1	-1	A^{2p+1}	1	-1	...	...	0	A^1	1	0
	0			0		A^{2p}	-1	0	A^{2p}	-1	0
						A^{2p+1}	1	-1	A^{2p+1}	1	-1
							0			0	

Figure 6: $a3\beta, \bar{c}^1, \dots, \bar{c}^p, \tilde{c}^1, \dots, \tilde{c}^{p-1}$

So, $\mathcal{S} = A_c^\perp$.

4 Conclusion

The set of solutions for (LP) is

$$\mathcal{S} = \text{co}\{x^k\}_{k \in K} + A_c^\perp,$$

with K finite, possibly empty. In particular, if $A_c^\perp = \{0_{\mathbb{R}^n}\}$, then $\mathcal{S} = \text{co}\{x^k\}_{k \in K}$. Each element x^k , as optimal solution, can be found by the simplex algorithm and the cardinal of K is most equal to the number of pivoting operations (non-repetitive, avoiding degeneracy),

$$\dim A_c^\perp = \text{card } \overline{\mathcal{B}}_0^\beta,$$

where $\overline{\mathcal{B}}_0^\beta = \overline{\mathcal{B}}_{0-} \cup \{\bar{i} \in \overline{\mathcal{B}}_{0+} \mid \overline{\mathcal{B}}_{0-} \neq \emptyset\}$, with $\overline{\mathcal{B}}_{0+} = \{\bar{i} \in \overline{\mathcal{B}}_0 \mid \exists k \in \mathcal{B} \text{ with } \alpha_{ik} > 0\}$.

Note that (LP) includes the system of linear inequalities

$$(SLI) \quad A \cdot x^T \leq b^T,$$

since each component of $x \in \mathbb{R}^n$, $x_j \in \mathbb{R}$ can be written $x_j = x'_j - x''_j$, with $x'_j, x''_j \geq 0$. More, it is equivalent to (LP)

$$\begin{cases} 0 \longrightarrow \min \\ A \cdot (x_j - x''_j)_{1 \leq j \leq n}^T \leq b^T \\ x'_j, x''_j \geq 0, 1 \leq j \leq n. \end{cases}$$

The first restriction $\sum_{j=1}^n a_{1j}x_j + x_{n+1} - b_1 = 0$ can be considered as objective function

$$\begin{cases} \sum_{j=1}^n a_{1j}x_j + x_{n+1} - b_1 \longrightarrow \min \\ \sum_{j=1}^n a_{ij}(x'_j - x''_j) + x_{n+i} = b_j, 1 \leq i \leq m \\ x'_j, x''_j, x_{n+i} \geq 0, 1 \leq i \leq m, 1 \leq j \leq n, \end{cases}$$

therefore simplex algorithm 3.0 can be applied to (SLI) and determine $\mathcal{S}$. To exemplify, one can go through Example 2.1 and Example 3.3. Note that, for an arbitrary index set the best approximate solutions of (SLI) are characterized in [7]. Also, $\mathcal{S}$ can be a feasible set to a certain subproblem [5].

By the same algorithm boundedness of $\mathcal{S}$ can be determined.

Immediate possible extensions are integer linear programming, matrix games, and vector linear programming.

The generalized simplex method given in [6], as a generalized pivoting

algorithm type, could be generalized for an arbitrary S , i.e. bounded or not. More precisely, Theorem 3.1 that indicates the optimal face could be adapted. The proof of Theorem 3.2 is based on the hypothesis that S is bounded.

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