

Quantitative Voronovskaya-type theorems for Jackson operators

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ABSTRACT. For some Jackson-Matsuoka operators quantitative Voronovskaya type theorems are presented.

KEY WORDS: Jackson-Matsuoka operators, quantitative Voronovskaya type theorems.

MSC 2010: 41A36, 41A25, 42A10.

1 Introduction

In recent times there have been some interests in studying quantitative Voronovskaya-type theorems, but almost all the papers concern with positive linear operators in spaces of non-periodic functions. The methods used in those papers are not useful in dealing with periodic functions for two reasons (at least). First they use different kinds of Taylor's formula and second, in the non-periodical case do not appear conjugate functions.

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It is known that the Voronovskaya-type theorems are related with the saturation class of some families of operators. We have noticed that in the case of trigonometric polynomial approximation process the Voronovskaya-type theorems depend on particular properties of the operators. In this paper we consider the Jackson operators (other approximation methods will be studied in forthcoming papers).

Let $C_{2\pi}$ denote the Banach space of all 2π -periodic, continuous functions f defined on the real line $\mathbb{R}$ with the sup norm

$$\|f\|_{\infty} = \max_{x \in [-\pi, \pi]} |f(x)|.$$

For $1 \leq p < \infty$, the Banach space $\mathbb{L}^p$ consists of all 2π -periodic, p -th power Lebesgue integrable functions f on $\mathbb{R}$ with the norm

$$\|f\|_p = \left(\frac{1}{2\pi} \int_{-\pi}^{\pi} |f(x)|^p dx \right)^{1/p}.$$

In order to simplify we write $X^p = \mathbb{L}^p$ for $1 \leq p < \infty$ and $X^{\infty} = C_{2\pi}$. by W_p^r we mean the family of all functions $f \in X^p$ such that $f, \dots, D^{r-1}(f)$ are absolutely continuous and $D^r(f) \in X^p$. Here $D(f) = D^1(f) = f'$ and $D^{r+1}(f) = D(D^r(f))$.

Recall that for $f \in \mathbb{L}^1$ and $k \in \mathbb{N}_0$, the Fourier coefficients are defined by

$$a_k(f) = \frac{1}{\pi} \int_{-\pi}^{\pi} f(t) \cos(kt) dt \quad \text{and} \quad b_k(f) = \frac{1}{\pi} \int_{-\pi}^{\pi} f(t) \sin(kt) dt,$$

and the (formal) Fourier series is given by

$$f(x) \sim \frac{a_0(f)}{2} + \sum_{k=1}^{\infty} (a_k(f) \cos(kx) + b_k(f) \sin(kx)) = \sum_{k=0}^{\infty} A_k(f),$$

For $r \in \mathbb{N}$, a function $f \in X^p$, and $h > 0$, the usual modulus of continuity smoothness of order r of f is defined by

$$(1.1) \quad \omega_r(f, t)_p = \sup_{|h| \leq t} \|(I - T_h)^r(f)\|_p,$$

where $T_h(f, x) = f(x + h)$ is the translation operator. We also use the notations $\Delta_h^r f(x) = (I - T_h)^r(f)$.

For $r, n \in \mathbb{N}$, $x \in \mathbb{R}$, and $f \in X^1$, the Jackson-Matsuoka operator is defined by

$$(1.2) \quad \mathbb{J}_{rn-r}(f, x) = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x+t) J_{rn-r}(t) dt,$$

where

$$J_{rn-r}(t) = \frac{1}{\gamma_{rn-r}} \left(\frac{\sin(nt/2)}{\sin(t/2)} \right)^{2r},$$

and γ_{rn-r} is chosen from the condition

$$\frac{1}{\gamma_{rn-r}} \left(\frac{\sin(nt/2)}{\sin(t/2)} \right)^{2r} = \frac{1}{2} + \sum_{k=1}^{r(n-1)} \mu_{k,n,r} \cos(kt).$$

The main purpose of the paper is to present quantitative Voronovskaya-type theorem for the operators $\mathbb{J}_{2n-2}$ and $\mathbb{J}_{3n-3}$. These tasks will be accomplished in Section 3. In Section 2 we collect some known facts and present a general result for trigonometric polynomial positive linear operators (see Theorem 1).

2 General results for positive polynomials

The convolution of two functions f and g in $\mathbb{L}^1$, g an even function, is defined by

$$(f * g)(x) = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x+t) g(t) dt = \frac{1}{\pi} \int_0^{\pi} (f(x+t) + f(x-t)) g(t) dt.$$

The main result of this section is Theorem 2.1 below.

In this section we fix a kernel

$$(2.1) \quad Q_n(x) = \frac{1}{2} + \sum_{k=1}^n \lambda_{k,n} \cos(kx) \geq 0,$$

The algebraic moment of order 2 is defined by

$$(2.2) \quad M_2(Q_n) = \frac{1}{\pi} \int_0^\pi v^2 Q_n(v) dv.$$

The estimate in Proposition 2.1 appeared in different places (see [5, Lemma 1] and [4, Lemma 1.5.7]).

Proposition 2.1 *If Q_n is given by (2.1), then*

$$2(1 - \lambda_{1,n}) \leq M_2(Q_n) \leq \frac{\pi^2}{2}(1 - \lambda_{1,n})$$

and

$$M_4(Q_n) \leq \frac{\pi^4}{8} \left(2(1 - \varrho_{1,n}) - \frac{1}{2}(1 - \varrho_{2,n}) \right).$$

The following proposition seen to be known, we include a proof for the benefit of the reader.

Proposition 2.2 *Assume $1 \leq p \leq \infty$. Let $Q_n(t)$ be given by (2.1) and set $K_n(f) = (f * Q_n)$. If $g \in W_p^2$, then*

$$\|g - K_n(g)\|_p \leq \frac{\pi^2}{2}(1 - \lambda_{1,n})\|D^2(g)\|_p.$$

Proof. Since Q_n is an even function

$$\begin{aligned} K_n(f, x) - f(x) &= K_n(f, x) - \frac{2}{\pi} \int_0^\pi f(x) Q_n(t) dt \\ &= \frac{1}{\pi} \int_0^\pi \left(f(x+t) - 2f(x) + f(x-t) \right) Q_n(t) dt, \end{aligned}$$

If we set $\Delta_t^2 g(x) = g(x+t) - 2g(x) + g(x-t)$, using the generalized Minkowski

inequality we obtain

$$\begin{aligned}
\|g - K_n(g)\|_p &\leq \frac{1}{\pi} \int_0^\pi Q_n(t) \left(\frac{1}{2\pi} \int_0^\pi |\Delta_t^2 g(x)|^p dx \right)^{1/p} dt \\
&\leq \|D^2(g)\|_p \frac{1}{\pi} \int_0^\pi t^2 Q_n(t) dt = M_2(Q_n) \|D^2(g)\|_p \\
&\leq \frac{\pi^2}{2} (1 - \lambda_{1,n}) \|D^2(g)\|_p
\end{aligned}$$

where we use Proposition 2.1. $\square$

Theorem 2.1 Assume $1 \leq p \leq \infty$, and the sequence $\{Q_n\}$ is given by (2.1). If $K_n(f) = f * Q_n$, then for each $g \in W_p^2$ one has

$$\begin{aligned}
\left\| \frac{K_n(g) - g}{\Lambda_n} - \frac{D^2(g)}{2} \right\|_p &\leq \left(1 + \frac{\pi^3}{8\sqrt{2}\sqrt{1 - \lambda_{1,n}}} \sqrt{4 - \frac{1 - \lambda_{2,n}}{1 - \lambda_{1,n}}} \right) \\
&\quad \times \omega_1 \left(D^2(g), \sqrt{1 - \lambda_{1,n}} \right)_p,
\end{aligned}$$

where $\Lambda_n = M_2(Q_n)$ is the algebraic moment defined by (2.2).

Proof. We present a proof for the $L_{2\pi}^p$ spaces, $1 < p < \infty$.

For $x, v \in [-\pi, \pi]$ set

$$\Theta_{x,v}(s) = \begin{cases} \chi_{[x, x+v]}(s), & v \geq 0, \\ \chi_{[x+v, x]}(s), & v < 0, \end{cases}$$

where $\chi_A(s)$ is the characteristic function of the set A .

Since

$$\begin{aligned}
g(x+v) &= g(x) + D^1(g, x)v + \int_x^{x+v} (x+v-s) D^2(g, s) ds, \\
\frac{1}{\pi} \int_{-\pi}^\pi v Q_n(v) dv &= \frac{1}{\pi} \int_0^\pi v Q_n(v) dv + \frac{1}{\pi} \int_0^\pi (-v) Q_n(-v) dv = 0,
\end{aligned}$$

because Q_n is an even trigonometric polynomial, and

$$\int_x^{x+v} (x+v-s) ds = \frac{v^2}{2},$$

one has

$$\begin{aligned}
& \frac{1}{\pi} \int_{-\pi}^{\pi} g(x+v) Q_n(v) dv - g(x) - \frac{D^2(g, x)}{2\pi} \int_{-\pi}^{\pi} v^2 Q_n(v) dv \\
&= \frac{1}{\pi} \int_{-\pi}^{\pi} \left(g(x+v) - g(x) - D^2(g, x) \frac{v^2}{2} \right) Q_n(v) dv \\
&= \frac{1}{\pi} \int_{-\pi}^{\pi} \left(g(x+v) - g(x) - D^1(g, x)v - D^2(g, x) \frac{v^2}{2} \right) Q_n(v) dv \\
&= \frac{1}{\pi} \int_{-\pi}^{\pi} \left(\int_x^{x+v} (x+v+s) D^2(g, s) - D^2(g, x)(x+v+s) ds \right) Q_n(v) dv \\
&= \frac{1}{\pi} \int_{-\pi}^{\pi} \left(\int_x^{x+v} (x+v+s) (D^2(g, s) - D^2(g, x)) ds \right) Q_n(v) dv \\
&= \frac{1}{\pi} \int_{-\pi}^{\pi} \left(\int_x^{x+v} (x+v+s) (D^2(g, s) - D^2(g, x)) ds \right) Q_n(v) dv \\
&= \frac{1}{\pi} \int_{-\pi}^{\pi} \left(\text{sign}(v) \int_{-\pi}^{\pi} \Theta_{x,v}(s) (x+v+s) (D^2(g, s) - D^2(g, x)) ds \right) Q_n(v) dv.
\end{aligned}$$

Therefore

$$\begin{aligned}
& \int_{-\pi}^{\pi} \left| \frac{1}{\pi} \int_{-\pi}^{\pi} g(x+v) Q_n(v) dv - g(x) - \frac{D^2(g, x)}{2\pi} \int_{-\pi}^{\pi} v^2 Q_n(v) dv \right|^p dx \\
&\leq \int_{-\pi}^{\pi} \left| \frac{1}{\pi} \int_{-\pi}^{\pi} \left(\int_{-\pi}^{\pi} \Theta_{x,v}(s) (x+v-s) (D^2(g, s) - D^2(g, x)) ds \right) Q_n(v) dv \right|^p dx.
\end{aligned}$$

If

$$F(x, v) = \int_{-\pi}^{\pi} \left| \Theta_{x,v}(s) (x+v-s) (D^2(g, s) - D^2(g, x)) \right| ds,$$

it follows from the generalized Minkowski inequality that

$$\begin{aligned}
& \pi \|K_n(f) - f - (\Lambda_n/2) D^2(g)\|_p \leq \int_{-\pi}^{\pi} Q_n(v) \left(\frac{1}{2\pi} \int_{-\pi}^{\pi} (F(x, v))^p dx \right)^{1/p} dv \\
&= \int_0^{\pi} Q_n(v) \left(\left(\frac{1}{2\pi} \int_{-\pi}^{\pi} (F(x, v))^p dx \right)^{1/p} + \left(\frac{1}{2\pi} \int_{-\pi}^{\pi} (F(x, -v))^p dx \right)^{1/p} \right) dv.
\end{aligned}$$

But, if $0 \leq v \leq \pi$, we use again the generalized Minkowski inequality to obtain

$$\begin{aligned}
 & \left(\frac{1}{2\pi} \int_{-\pi}^{\pi} (F(x, v))^p dx \right)^{1/p} \\
 &= \left(\frac{1}{2\pi} \int_{-\pi}^{\pi} \left(\int_x^{x+v} \left| (x+v-s) (D^2(g, s) - D^2(g, x)) \right| ds \right)^p dx \right)^{1/p} \\
 &= \left(\frac{1}{2\pi} \int_{-\pi}^{\pi} \left(\int_0^v \left| (v-w) (D^2(g, x+w) - D^2(g, x)) \right| dw \right)^p dx \right)^{1/p} \\
 &\leq \int_0^v (v-w) \left(\frac{1}{2\pi} \int_{-\pi}^{\pi} |D^2(g, x+w) - D^2(g, x)|^p dx \right)^{1/p} dw \\
 &\leq \int_0^v (v-w) \omega_1(g'', w)_p dw = \frac{1}{2} v^2 \omega_1(D^2(g), v)_p.
 \end{aligned}$$

A similar inequality follows if $-\pi \leq v < 0$. Therefore, taking into account (2.2), and Proposition 2.1, we obtain

$$\begin{aligned}
 & \|K_n(f) - f - (\Lambda_n/2)D^2(g)\|_p \leq \frac{1}{\pi} \int_0^{\pi} Q_n(v) v^2 \omega_1(D^2(g), v)_p dv \\
 &\leq \omega_1(D^2(g), \sqrt{1-\lambda_{1,n}})_p \frac{1}{\pi} \int_0^{\pi} Q_n(v) v^2 \left(1 + \frac{v}{\sqrt{1-\lambda_{1,n}}} \right) dv \\
 &\leq \omega_1(D^2(g), \sqrt{1-\lambda_{1,n}})_p \left(\Lambda_n + \frac{1}{\sqrt{1-\lambda_{1,n}}} \sqrt{M_2(Q_n)M_4(Q_n)} \right) \\
 &\leq \omega_1(D^2(g), \sqrt{1-\lambda_{1,n}})_p \left(\Lambda_n + \sqrt{\frac{\pi^2}{2} \frac{\pi^4}{8} \left(2(1-\lambda_{1,n}) - \frac{1}{2}(1-\lambda_{2,n}) \right)} \right) \\
 &= \omega_1(D^2(g), \sqrt{1-\lambda_{1,n}})_p \left(\Lambda_n + \frac{\pi^3 \sqrt{(1-\lambda_{1,n})}}{4\sqrt{2}} \sqrt{4 - \frac{1-\lambda_{2,n}}{1-\lambda_{1,n}}} \right)
 \end{aligned}$$

We have proved that

$$\begin{aligned}
& \left\| \frac{1}{\Lambda_n} (K_n(g) - g) - \frac{1}{2} D^2(g) \right\|_p \\
& \leq \left(1 + \frac{\pi^3}{4\sqrt{2}} \frac{\sqrt{1-\lambda_{1,n}}}{\Lambda_n} \sqrt{4 - \frac{1-\lambda_{2n}}{1-\lambda_{1,n}}} \right) \omega_1 \left(D^2(g), \sqrt{1-\lambda_{1,n}} \right)_p \\
& \leq \left(1 + \frac{\pi^3}{8\sqrt{2}} \frac{1}{\sqrt{1-\lambda_{1,n}}} \sqrt{4 - \frac{1-\lambda_{2n}}{1-\lambda_{1,n}}} \right) \omega_1 \left(D^2(g), \sqrt{1-\lambda_{1,n}} \right)_p. \quad \square
\end{aligned}$$

Remark 2.1 For a sequence $\{Q_n\}$ of even positive trigonometric polynomials, Theorem 2.1 is interesting only in the case $\lambda_{1,n} \rightarrow 1$ as $n \rightarrow \infty$ and

$$(2.3) \quad \sup_{n \in \mathbb{N}} \frac{1}{\sqrt{1-\lambda_{1,n}}} \sqrt{4 - \frac{1-\lambda_{2n}}{1-\lambda_{1,n}}} < \infty.$$

3 Voronovskaya-type theorems

For the operators J_{2n-2} some Voronovskaya type relations are known. For instance, Korovkin [7] proved that

$$\lim_{n \rightarrow \infty} \|2n^2(J_{2n-2}(f) - f) - 3D^2(f)\|_\infty = 0.$$

On the other hand, if $f \in C_{2\pi}$ and $D^2(f, x)$ exists as a point x , then

$$\lim_{n \rightarrow \infty} (n+1)^2 (\mathbb{J}_{2n-2}(f, x) - f(x)) = \frac{3}{2} D^2(f, x).$$

(see [8, page 134] and [2, Theorem 6.2]) and if $f \in W_p^2$, $1 < p < \infty$, then

$$(3.1) \quad \lim_{n \rightarrow \infty} \left\| n^2 (\mathbb{J}_{2n-2}(f) - f) - \frac{3}{2} D^2(f) \right\|_p = 0,$$

(see [3, page 387]).

Point-wise Voronovskaya-type theorems for the Jackson-Matsuoka operators $\mathbb{J}_{rn-r}$ were provided by Schurer [11, Theorem 22] for $f \in C_{2\pi}^2$

and Rathore [10, Lemma 3] (not in a quantitative form). For X^p spaces there are less exact estimates in [6, page 160]. For the general case a result was given by Matsuoka in [9].

In this section we prove quantitative Voronovskya type theorems for the operators $\mathbb{J}_{2n-2}$ and $\mathbb{J}_{3n-3}$. In particular we show, with other arguments, that the assertion (3.1) can be improved. In studying both operators some differences appear. In order to explain it in a clear form, let us recall some known facts.

Proposition 3.1 (see [12, Theorem 1]) *For each $n \geq 3$, there exists reals $\varrho_{k,n}$, $1 \leq k \leq 2n-2$ such that*

$$\frac{1}{\gamma_{2n-2}} \left(\frac{\sin(nt/2)}{\sin(t/2)} \right)^4 = \frac{1}{2} + \sum_{k=1}^{2n-2} \varrho_{k,n} \cos(kt),$$

In particular

$$\frac{1}{\gamma_{2n-2}} = \frac{3}{2(2n^3 + n)}, \quad \varrho_{1,n} = 1 - \frac{3}{2n^2 + 1}, \quad \text{and} \quad \varrho_{2,n} = 1 - \frac{3(4n-3)}{n(2n^2 + 1)}.$$

It follows from the previous proposition that for the operator $\mathbb{J}_{2n-2}$ one has

$$\frac{1}{1 - \varrho_{1,n}} \left(4 - \frac{1 - \varrho_{2,n}}{3(1 - \varrho_{1,n})} \right) = \frac{(2n^2 + 1)}{3} \frac{3}{n}.$$

Hence, the condition in equation (2.3) does not hold and Theorem 2.1 is not useful in this case.

Let $\mathbb{T}_n$ denotes the family of all real trigonometric polynomials. For a polynomial $T(x) = a_0/2 + \sum_{k=1}^n (a_k \cos(kx) + b_k \sin(kx))$, we set

$$\tilde{T}(x) = \sum_{k=1}^n (b_k \cos(kx) - a_k \sin(kx)).$$

For $n \in \mathbb{N}$ and $f \in X^1$ the Fejér sum of order n is defined by

$$\sigma_n(f, x) = \sum_{k=0}^n \left(1 - \frac{k}{n+1} \right) A_k(f, x).$$

Notice that, if $T \in \mathbb{T}_n$, then

$$(3.2) \quad T - \sigma_n(T) = -\frac{D(\tilde{T})}{n+1}.$$

For $1 \leq p \leq \infty$ and $f \in X^p$, the best approximation of f by elements of $\mathbb{T}_n$ is defined by

$$E_n(f)_p = \inf_{T \in \mathbb{T}_n} \|f - T\|_p.$$

The proof of the following result will appear in [1].

Theorem 3.1 *Assume $1 \leq p \leq \infty$, $r, n \in \mathbb{N}$ and $n \geq \max\{2, r\}$.*

(i) *If $g \in W_p^r$, $T \in \mathbb{T}_n$, and $\|g - T\|_p = E_{n,p}(g)$, then*

$$\|D^r(g) - D^r(T)\|_p \leq \left(4 + \frac{1}{r} + \ln(2r)\right) \left(1 + \frac{e\pi}{2}\right) E_{n,p}(D^r(g)).$$

We will use Theorem 3.1 in the particular case $r = 2$. In order to simplify the assertion a bigger constant is convenient. That is

$$\left(\frac{9}{2} + \ln 4\right) \left(1 + \frac{e\pi}{2}\right) \leq (5.9)(5.27) < 31.05.$$

Theorem 3.2 *Assume $1 \leq p \leq \infty$. If $f \in W_p^2$ and $n \in \mathbb{N}$, $n \geq 3$, then*

$$\begin{aligned} \|(2n^2 + 1)(\mathbb{J}_{2n-2}(f) - f) - 3D^2(f)\|_p &\leq 4E_{n,p}(f) + (219 + 47\pi^2)E_{n,p}(D^2(g)) \\ &\quad + 2\|(I - \sigma_n)(f)\|_p + 2\|(I - \sigma_n)(D^2(f))\|_p, \end{aligned}$$

where σ_n is the Fejér operator.

Proof. For each $n \in \mathbb{N}$, chose $T \in \mathbb{T}_n$ such that $E_{n,p}(f) = \|f - T\|_p$. Notice that

$$\begin{aligned} (2n^2 + 1)(\mathbb{J}_{2n-2}(f) - f) - 3D^2(f) &= (2n^2 + 1)(\mathbb{J}_{2n-2}(T) - T) - 3D^2(f) \\ &\quad + (2n^2 + 1)(\mathbb{J}_{2n-2}(f - T) - (f - T)) + 3(D^2(T) - D^2(f)). \end{aligned}$$

We will estimate each term separately.

It can be proved that, for each $n \in \mathbb{N}$ and $T \in \mathbb{T}_n$ one has

$$\mathbb{J}_{2n-2}(T) - T - \frac{3}{2n^2+1}D^2(T) = -\frac{3}{2n(2n^2+1)}D^1(\tilde{T}) + \frac{3}{2n(2n^2+1)}D^3(\tilde{T}).$$

Taking into account (3.2) we can write

$$\begin{aligned} & \mathbb{J}_{2n-2}(T) - T - \frac{3}{2n^2+1}D^2(T) \\ &= \frac{3(n+1)(\sigma_n - I)(T)}{2n(2n^2+1)} + \frac{3(n+1)}{2n(2n^2+1)}(I - \sigma_n)(D^2(T)), \end{aligned}$$

where σ_n is de Fejér operator.

If $n \geq 3$, then

$$\frac{3}{2}\left(1 + \frac{1}{n}\right) \leq 2.$$

Hence

$$\begin{aligned} & \|(2n^2+1)(\mathbb{J}_{2n-2}(T) - T) - 3D^2(T)\|_p \leq 2\|(\sigma_n - I)(T)\|_p + 2\|(I - \sigma_n)(D^2(T))\|_p \\ & \leq 4\|T - f\|_p + 2\|(\sigma_n - I)(f)\|_p + 2\|(I - \sigma_n)(D^2(f))\|_p + 4\|D^2(T - f)\|_p \\ & \leq 4E_{n,p}(f) + 2\|(\sigma_n - I)(f)\|_p + 2\|(I - \sigma_n)(D^2(f))\|_p + 125E_{n,p}(D^2(f)). \end{aligned}$$

On the other hand, since $f - T \in W_p^2$, as a direct consequence of Proposition 2.2 (recall that $\varrho_{1,n} = 1 - 3/(2n^2+1)$) we know that

$$\begin{aligned} & (2n^2+1)\|\mathbb{J}_{2n-2}(f - T) - (f - T)\|_p \leq \frac{3\pi^2}{2}\|D^2(f - T)\|_p \\ & \leq \frac{3\pi^2}{2}(31.05)E_{n,p}(D^2(f)) \leq 46.7\pi^2E_{n,p}(D^2(f)), \end{aligned}$$

where we use again Theorem 3.1.

The term $3(D^2(T) - D^2(f))$ can also be estimate as in Theorem 3.1. That is

$$\|3(D^2(T) - D^2(f))\|_p \leq 3 \times (31.05)E_{n,p}(D^2(f)) \leq 94E_{n,p}(D^2(f)).$$

In conclusion we have proved that

$$\begin{aligned} \|(2n^2 + 1)(\mathbb{J}_{2n-2}(f) - f) - 3D^2(f)\|_p &\leq 4E_{n,p}(f) + (219 + 47\pi^2)E_{n,p}(D^2(g)) \\ &\quad + 2\|(I - \sigma_n)(f)\|_p + 2\|(I - \sigma_n)(D^2(f))\|_p. \end{aligned} \quad \square$$

We use Proposition 3.2 to derive a Voronovskaya type theorem for the operators $\mathbb{J}_{3n-3}$ via Theorem 2.1.

Proposition 3.2 (see [12, Theorem 1]) *For each $n \geq 3$, there exists reals $q_{k,n}$, $1 \leq k \leq 3n - 3$ such that*

$$\frac{1}{\gamma_{3n-3}} \left(\frac{\sin(nt/2)}{\sin(t/2)} \right)^6 = \frac{1}{2} + \sum_{k=1}^{3(n-1)} q_{k,n} \cos(kt).$$

Moreover

$$\begin{aligned} \gamma_{3n-3} &= \frac{n(11n^4 + 5n^2 + 4)}{10}, \\ q_{1,n} &= 1 - \frac{n(n^2 + 1)}{\gamma_{3n-3}}, \quad \text{and} \quad q_{2,n} = 1 - \frac{20(2n^2 - 1)}{9n^4 + 5n^2 + 4}. \end{aligned}$$

Proposition 3.3 *Let $q_{k,n}$ be the coefficients given in Proposition 3.2. For each $n \in \mathbb{N}$, one has*

$$\frac{1}{1 - q_{1,n}} \left(4 - \frac{1 - q_{2,n}}{1 - q_{1,n}} \right) \leq \frac{27}{5} \quad \text{and} \quad 1 - q_{1,n} \leq \frac{10}{n^2}.$$

Proof. For the representation of the convergence factors $q_{1,n}$ and $q_{2,n}$, one has

$$\begin{aligned} \frac{1}{1 - q_{1,n}} \left(4 - \frac{1 - q_{2,n}}{1 - q_{1,n}} \right) &= \frac{9n^4 + 5n^2 + 4}{10(n^2 + 1)} \left(4 - \frac{2n(2n^2 - 1)}{n(n^2 + 1)} \right) \\ &= \frac{6(9n^4 + 5n^2 + 4)}{10(n^2 + 1)^2} \leq \frac{27}{5} \end{aligned}$$

and

$$1 - q_{1,n,3} = \frac{n(n^2 + 1)}{\gamma_{3n-3}} = \frac{10n(n^2 + 1)}{n(9n^4 + 5n^2 + 4)} \leq \frac{10(n^2 + 1)}{n^4 + n^2} = \frac{10}{n^2}. \quad \square$$

Theorem 3.3 Assume $1 \leq p \leq \infty$ and $\mathbb{J}_{3n-3}$ is the Jackson-Matsuoka operator (1.2). For each $g \in W_p^2$ one has

$$\left\| \frac{1}{\Lambda_n} \left(\mathbb{J}_{3n-3}(g) - g \right) - \frac{1}{2} D^2(g) \right\|_p \leq 6\omega_1 \left(D^2(g), \frac{\sqrt{10}}{n} \right)_p,$$

where $\Lambda_n = M_2(J_{3n-3})$ see (2.2).

Proof. It follows from Theorem 2.1 and Proposition 3.3. $\square$

References

- [1] J. Bustamante and L. Flores-de-Jesús, Strong converse inequalities and quantitative Voronovskaya-type theorems for trigonometric Fejér sums, to appear.
- [2] P. L. Butzer, Representation and approximation of functions by general singular integrals, *Nederl. Akad. Wetensch. Proc. Series*, 63 (22) (1960), 1-24.
- [3] P. L. Butzer and E. Görlich, Saturationsklassen und asymptotische Eigenschaften trigonometrischer singulärer Integrale, *Festschrift zur Gedchtnisfeier für Karl Weierstraß 1815-1965 Arbeitsgemeinschaft für Forschung des Landes Nordrhein-Westfalen*, Bd. 33, Köln (1966), 339 -392.
- [4] P. L. Butzer, R. J. Nessel, *Fourier Analysis and Approximation*, New York-Base1 (1971).
- [5] E. Görlich and E. L. Stark, Über beste Konstanten und asymptotische Entwicklungen positiver Faltungsintegrale und deren Zusammenhang mit dem Saturationsproblem, *Jber. Deutsch. Math.-Verein*, 72 (1970), 18-61.
- [6] H-B. Knoop and X. Zhou, Inequalities for trigonometric polynomials and some integral means, *J. Math. Anal. Appl.*, 255 (2001), 147-162.

- [7] P. P. Korovkin, On the order of the approximation of functions by linear positive operators, *Dokl. Akad. Nauk SSSR*, 114 (6) (1957), 1158-1161.
- [8] P. P. Korovkin, *Linear Operators and Approximation Theory*, Delhi (1960).
- [9] Y. Matsuoka, On the degree of approximation of functions by some positive linear operators, *Sci. Rep. Kagoshima Univ.*, 9 (1960), 11-16.
- [10] R. K. S. Rathore, On a sequence of linear trigonometric polynomial operators, *SIAM J. Math. Anal.*, 5 (1974), 908-917.
- [11] F. Schurer, *On Linear Positive Operators in Approximation Theory*, Mathematical Institute of the Technological University Delft, Report, 1962.
- [12] F. Schurer and F. W. Steutel, On linear positive operators of the Jackson type, *Mathematica*, 9 (32) (1967), 155-184.