

## Golden Ratio, a measure of beauty in art and science

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**ABSTRACT.** The Golden Ratio has been associated with the ideas of harmony and beauty in art, science and engineering. Many mathematical properties connected with it and its generalizations were studied along time. In this paper we present some aspects related to Golden Ratio as an aesthetic standard from the earliest times until now. We propose also a generalization of the Golden Ratio and applications.

**KEY WORDS:** aesthetic standards, golden ratio.

### 1 Introduction

The difficulty of defining a measure of the beauty of objects consists in the fact that the perception of beauty by an individual is subjective. On the other hand the notion of beauty differs from one area to another area in nature, art and science. Since ancient times there were numerous attempts

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to introduce aesthetic standards to quantify the notion of beauty. One of the most important standards used from ancient time until now is based on the Golden Ratio. In this paper we present some aspects related to the Golden Ratio, its evolution and its applications over the centuries. The first definition of the Golden Ratio was given around 300 BC by the famous founder of geometry, Euclid of Alexandria. Euclid defined a proportion derived from a simple division of a segment into what he called it "extreme and mean ratio." Recall the definition of Euclid: "A straight line is said to have been cut in extreme and mean ratio when, as the whole line is to the greater segment, so is the greater to the lesser". Let  $AB$  be a segment and  $C$  a point belonging to it such that  $CB > CA$  (Fig. 1). If

$$\frac{AC}{CB} = \frac{CB}{AB} \quad \text{or} \quad CB = \sqrt{AB \cdot AC}$$

then the segments  $CB$  and  $CA$  are said to be in the Golden Ratio. Observe that the Golden Ratio is defined by the geometric mean. The number  $\varphi = \frac{CB}{AC} = \frac{1+\sqrt{5}}{2}$  is called the Golden Ratio and it is the positive solution of the following equation

$$t^2 - t - 1 = 0.$$

The approximate value of the golden ratio, which is an irrational number, is

$$\varphi = \frac{1 + \sqrt{5}}{2} = 1.61803399 \dots$$



Figure 1

The number  $\varphi$  is also known as the "Golden Number", "Golden Section", "Golden Ratio", "Divine Proportion", or "Extreme and Mean Ratio". The term "Divina Proportione" was the title of Pacioli's book published in 1509. We can find the term Golden Section as "proportionally divided" in Clavius 1574 edition of "The Elements". Kepler, in his "Mysterium

Cosmographycum” used the term ”proportional division” for the Division in Extreme and Mean Ratio (DEMR). The term ”Medial Section” was used by Leslie (1820) in connection with the DEMR. The notion of ”Golden Section” (”Sectio Aurea”) takes its origin from Claudia Ptolemey who gave this name to the number 1.618, after he had been convinced that the aspect of a person with perfect constitution is divided naturally in this ratio. The given name became popular due to Leonardo da Vinci who often used it.

## 2 Golden Ratio and the idea of Harmony

### 2.1 Golden Ratio in Ancient Greece

According to the ancient Greeks, *harmony means what is the opposite of chaos and this is really the essence of the organization and existence of the Universe*. As stated in [14] it seems that sometimes a return to positive aspects of ”sacral geometry” and ”sacral knowledge” contained in the Talmud, Bible and Chinese ”Book of Changes” and a rethinking of religious principles is beneficial for the current stage of development of science. Sacral Geometry is a way of perceiving the Universe by Man. Pythagoras considered that it is ”the hidden science of the gods”. The basic idea of sacral doctrines is the following: *The Universe exists as a harmonious and proportionate entity. Harmony is the basis of beauty (aesthetics)*.

- The number  $\pi = (\text{circle length}) : (\text{diameter}) \approx 3.14$
- The number  $\sqrt{2} = (\text{square diagonal}) : (\text{square side}) \approx 1.414$
- The number  $\sqrt{3} = (\text{cube diagonal}) : (\text{cube side}) \approx 1.732$
- The number  $\sqrt{5} \approx 2.236 = \text{diagonal of double square}$
- The number  $\varphi = \frac{1+\sqrt{5}}{2} \approx 1.618$  (golden section or golden ratio)

The first two constants,  $\pi$  and  $\sqrt{2}$ , appear in many proportions of nature or human creations, linked to circle and square. Pythagoreans considered  $\sqrt{5}$  a sacral number and the double square appears in many ancient creations as: the Chamber of Pharaohs, the Great Pyramid, and the Temple of Solomon.

The number  $\sqrt{3}$  is linked not only to the cube but also to the so-called geometrical "vesica piscis" which is the intersection of two equal circles each containing the center of the other. If the circles have unit radius, then the common chord is  $\sqrt{3}$ . This number was commonly used by artists and architects in the Renaissance. Also it appears as the ratio of the diagonal and the side of a regular pentagon. All constants of "sacral geometry" appear in some ratios of regular polyhedral elements. Pythagoras showed that there are only five regular polyhedral in three-dimensional space, called "Platonic Solids".

The Platonic Solids are the *tetrahedron* (with four triangular faces), the *cube* (with six square faces), the *octahedron* (with eight equilateral triangles), the *dodecahedron* (with twelve pentagonal faces), and the *icosahedron* (with twenty triangular faces). Platonic Solids are the source of many ideas and concepts of modern science. They were used by Plato in his theory of the Cosmos. Basic elements of his theory are: Fire, Air, Water, and Earth. The atoms of these elements have the form of Platonic Solids:

- *fire = regular tetrahedron,*
- *air = octahedron,*
- *earth = cube,*
- *water = icosahedron.*

The dodecahedron was considered by Plato the most important figure in the Universe and it symbolizes the whole Universe and the wisdom in the Universe (see [6], [9], [17]).

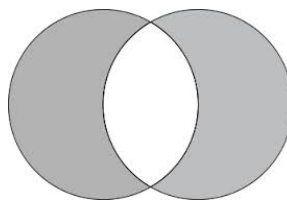


Figure 2: Vesica Piscis



Figure 3: Platonic Solids: tetrahedron, hexahedron (cube), octahedron, dodecahedron, icosahedron

Pythagoras is one of the most celebrated people in the history of science. The Pythagorean doctrine focused upon harmony, geometry, number theory, astronomy, and other topics, but the Pythagoreans, most of all, appreciated the results that were obtained in the theory of harmony because they confirmed their idea that *numbers determine everything*. Thanks to Pythagoras the Greek science could obtain new knowledge in the fields of philosophy, mathematics, and natural sciences.

The idea of harmony based on the Golden Ratio became one of the most fruitful ideas in the Greek art. *To take the subject and eliminate all that is superfluous is the plan of the ancient Greek artist.* This is the main idea of Greek art, where the Golden Section is the center of the canon of aesthetics. Pythagoras was the first who attempted mathematically to understand the essence of musical or harmonic ratios or intervals. Pythagoras knew that the intervals of the octave can be expressed by natural numbers, which fit the corresponding oscillations of the string, and these numerical relations were put forth by Pythagoras as the basis of his theory of musical harmony (see [17]).

The famous Greek philosopher Aristotle proposed order, proportionality, and size limitation. The order arises when certain relations and proportions hold between the whole and its parts. In music, Aristotle recognized the octave as the most beautiful consonance; he considered that the ratio of oscillations between the basic tone and its octave is expressed by the first natural numbers  $1 : 2$ . He also maintained that in poetry, the rhythmic relations of a verse are based on small numerical relations, and by this, a beautiful impression is expressed. The ancient Harmony theory, based on the principle of the division in extreme and mean ratio, became the "launching pad," upon which, subsequently, a harmony concept was developed in the

science and art of European culture (see [17]).

## 2.2 Golden Ratio in the Ancient Egypt

The cult of the dead pharaoh had great significance in the Egyptian religion. The monumental pyramids were constructed for the preservation of a pharaohs body, spirit, and for extolling his authority. Constructed by human hands, they deserve to be included among the Seven Wonders of the World. The pyramids had multiple functions. They served not only as vaults of a Pharaohs mortal remains, but they also were the attributes of his majesty, power, riches of country, the monuments of culture, the countrys history, and items of information about the pharaoh, his people, and life [4], [6], [17]. The Egyptians thought that the Golden Ratio was sacral, therefore it was very important in their religion. They used the Golden Ratio when building temples and places for the dead. If the proportions of their buildings weren't according to the Golden Ratio, the deceased might not make it to the afterlife or the temple would not be pleasing to the gods. As well, the Egyptians found the Golden Ratio to be pleasing to the eye. They used it in their system of writing and in the arrangement of their temples. The Egyptians were aware that they were using the golden ratio, but they called it the "sacral ratio." Remark that important elements of the Great Pyramid of Giza are closely related to Sacral Geometry:

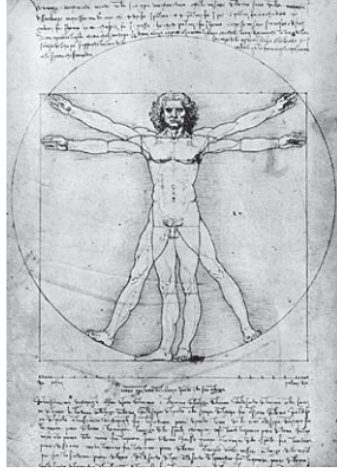
- the ratio of the apothem of the pyramid to the apothem of base obeys the Golden Ratio.
- the perimeter of its base equals that of a circle with a radius equal to its height.
- the dimensions of the Earth and Moon obey the same proportions as the dimensions of the Great Pyramid, and that they can be *directly* obtained using the Phythagorean 3 – 4 – 5 triangle.

### 2.3 Golden Ratio in Renaissance Epoch

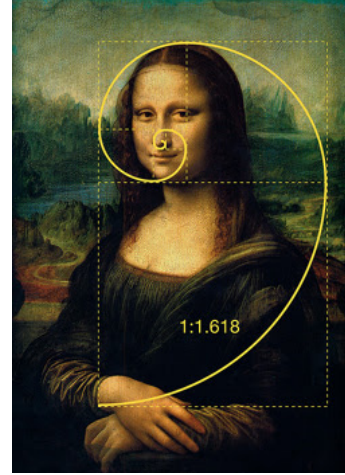
The Renaissance produced a significant change in direction in the history of the Golden Ratio. Now the Golden Ratio found its way into explanations of natural phenomena and into the arts. In the opinion of the American historian of mathematics, Morris Klein, the close merge of the religious doctrine about God as the creator of the Universe, and the antique idea of numerical harmony of the universe, became one of the major staples of the Renaissance culture [4], [5], [17]. Mainly, the objective of Renaissance science is presented in the following text from the well-known astronomer Johannes Kepler: *The overall objective of all researches of the external world should be to discover the rational order and harmony, which was embodied by the God in the Universe and then was presented for us by the God on mathematics language* (see [17]). Many of the assertions concerning the employment of the Golden Ratio in painting are directly associated with the presumed aesthetic properties of the Golden Rectangle. The "Ognissanti Madonna" also known as "Madonna in Glory, (currently in the Uffizi Gallery in Florence) is one of the greatest panel paintings by the famous Italian painter and architect Giotto di Bondone (1267 – 1337). The Madonna and Child are surrounded by angels and saints arranged in some sort of perspective "hierarchy". The whole paintings and their central figures are inscribed in a Golden Rectangle.

Another name that invariably turns up in almost every claim of the appearance of the Golden Ratio in art is that of Leonardo da Vinci. Some authors even attribute the invention of the name "the Divine Proportion" to Leonardo da Vinci. The ideal human figures, created by Leonardo da Vinci and Durer, are widely known. For a long time, the opinion existed that the "pentagonal" or "fivefold" symmetry, which is characteristic for flora and animals, is shown in the structure of the human body. The human body can be considered as a sample of the pentagram, where the human head, two hands, and two legs are, as if, beams of the pentagonal star. Such model found a reflection in the constructions of Leonardo da Vinci and Durer, in particular, in the figure of the well-known Vitruvian Man by Leonardo da Vinci.

*Mona Lisa.* Many specialists who analyzed the painting came to the conclusion that Leonardo da Vinci used the Golden Section precise and



(a) Vitruvian Man by Leonardo da Vinci



(b) Mona Lisa by Leonardo da Vinci

Figure 4

careful in his creation because it creates the impression that the Mona Lisa's eyes if they pursue the spectator moves around the painting. The main features of the golden ratio were highlighted below. Looking at the picture entirely, distance between your right and head Mona Lisa is 1,618 times the distance between its right finger and collarbone. The right side of the cheek is gold with small side compared to original golden rectangle. Another great scientist in Renaissance epoch was Luca Pacioli (1445 – 1514). Under the direct influence of Leonardo da Vinci, Luca Pacioli began to write the book "De Divina Proportione", published in 1509. The historical significance of this book is that it was the first mathematical book solely dedicated to the Golden Ratio. "De Divina Proportione" consists of three parts: the first part sets out the properties of the Golden Ratio, the second part is dedicated to the five regular polyhedrons, and the third part describes the applications of the Golden Section in architecture [12], [17].

## 2.4 Golden ratio in science

Harmony Mathematics can influence different fields of modern culture and become the source of new scientific, engineering, and cultural deriving forces. Their impact can be seen in the following branches of science (for more details see [4], [10, 11, 12, 13, 14], [17]):

- *Physics:* Many articles exhibit a substantial interest of modern theoretical physics and the physics of high energy particles in the Golden Section. The works of Shechtman, Butusov, Mauldin and William, El Naschie, and Vladimirov show that it is impossible to imagine the future progress in physical and cosmological research without the Golden Section. The famous Russian physicist Vladimirov from Moscow University finished his book "Metaphysics" with the following words:  
"Thus, it is possible to assert that in the theory of electroweak interactions there are relations that are approximately coincident with the Golden Section that play an important role in various areas of science and art" [15].
- *Philosophy.* "The Law of Structural Harmony of Systems" by Eduard Soroko [19] is possibly one of the most outstanding modern philosophical achievements. In essence Soroko's Law is a brilliant confirmation of the Generalized Golden Section Principle applied to self-organizing systems.
- *Botanic.* The new geometric theory of the botanic phyllotaxis phenomenon developed in the book "The Golden Section and Non-Euclidean geometry in nature and art" in 1994 by the Ukrainian architect Oleg Bodnar (Bodnars theory) is a brilliant confirmation of the hyperbolic Fibonacci and Lucas functions application in simulation of the phyllotaxis phenomenon.
- *Biology.* Cell growth is one of the actual biological problems. It was proven in [9] that binary cell division is regularly asymmetric and based on the Fibonacci p-numbers that yield from the Pascal triangle. It means that the binary cell division is based on the Generalized Principle of the Golden Section.

- *Medicine.* The Russian biologist Viktor Tsvetkov asserts in his book "Heart, the Golden Section and Symmetry" [19] that mammals cardiogram is subordinated to the Golden Section Principle. In Tsvetkovs opinion, the organization of the "golden" cardiac cycle is a result of a long evolution of mammals in the direction of optimization of their structure and functions.
- *Communication systems.* Fibonacci matrices and the "golden" matrices are the mathematical basis for a new coding theory [16], [18] that can be used effectively for error detection and correction in communication channels and for cryptographic protection of communication systems.
- *Probability theory.* It is impossible to overestimate the methodological importance of the deep mathematical connection of the Generalized Principle of the Golden Section to the Pascal triangle and binomial coefficients. It is clear that this connection can become the beginning of a reappraisal of many branches of modern mathematics and theoretical physics where combinatorial relations play an important role, in particular, in probability theory and statistical laws.

### 3 Generalizations of Golden Ratio and applications

Generalizations of Golden Ratio can be obtained by replacing the geometric mean from the definition of it by another mean. Recall first that a homogeneous mean is a function  $\mu : (0, \infty) \times (0, \infty) \rightarrow (0, \infty)$  with the properties

$$x < \mu(x, y) < y, \text{ for } 0 < x < y$$

and

$$\mu(\lambda x, \lambda y) = \lambda \mu(x, y), \text{ for every } \lambda, x, y \in (0, \infty).$$

More precisely, we say that  $x$  and  $y$  are in Generalized Golden Ratio (GGR) if satisfy the relation

$$(3.1) \quad y = \mu(x, x + y)$$

where  $\mu$  is a continuous, homogeneous mean ( see [1, 2, 7]). Since  $\mu$  is

homogeneous, equation (3.1) is equivalent to

$$(3.2) \quad \rho = \mu(1, 1 + \rho)$$

where  $\rho = \frac{y}{x}$  is the value of the Generalized Golden Ratio. Suppose in what follows that the equation (3.2) admits in the interval  $(1, \infty)$  a unique solution  $\rho_\mu$ . We present in what follows the value of the GGR for some particular homogeneous means (see [1, 2, 3]).

- **The arithmetic mean.** For  $a, b > 0$  consider the arithmetic mean

$$A(a, b) = \frac{a + b}{2}.$$

Suppose that the segment  $BC$  is the arithmetic mean of the segments  $AC$  and  $AB$ . Then, with the above notations, we have

$$y = A(x, x + y)$$

and the ratio  $\rho = \frac{y}{x}$  has the value  $\rho_A = 2$ .

- **The harmonic mean,** that is  $H(a, b) = \frac{2ab}{a+b}$ . If the two parts of the segment  $[AB]$  verify the relation

$$y = H(x, x + y),$$

then the ratio  $\rho = \frac{y}{x}$  satisfies the equation

$$\rho^2 - 2 = 0,$$

with the positive solution  $\rho_H = \sqrt{2}$ .

- **The quadratic mean.** For  $a, b > 0$

$$Q(a, b) = \sqrt{\frac{a^2 + b^2}{2}}$$

In this case from the relation  $y = Q(x, x + y)$  it follows the equation

$$\rho^2 - 2\rho - 2 = 0$$

with the positive solution  $\rho_Q = 1 + \sqrt{3}$ .

**The Golden Ratio** is considered to embody a **perfect proportion** between two quantities which seems to appear in different domains, such as the visual arts, architecture, design, philosophy, even science. In Ancient Egypt the Golden Ratio was used when building temples and places for the dead. If the proportions of their buildings weren't according to the Golden Ratio, the deceased might not make it to the afterlife or the temple would not be pleasing to the gods. As well, the Egyptians found the Golden Ratio to be pleasing to the eye.

Harmony and balance are still important factors in contemporary architecture as it was in the past. A building satisfying aesthetic standards is presented below (Fig. 5, Fig. 6).

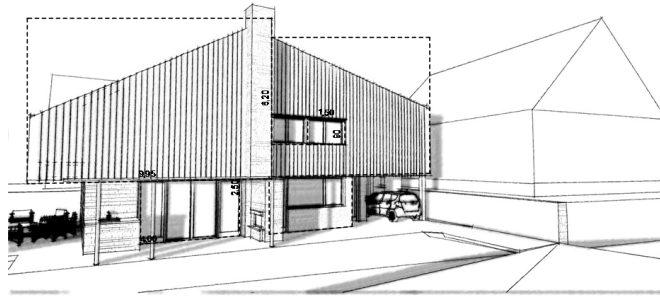


Figure 5

The blueprints or facades of numerous buildings were based on the Golden Ratio over the time such as Parthenon and Giza Pyramid. In Fig.5 the Golden Ratio is used to create proportions of the entire building, but also for elements of the facade, such as windows and doors. Architecture is represented through the medium of drawing, so mostly lines. In this example, The Golden Rectangle is used to create the entire building, which means that the proportions between the two lengths of the rectangle equals

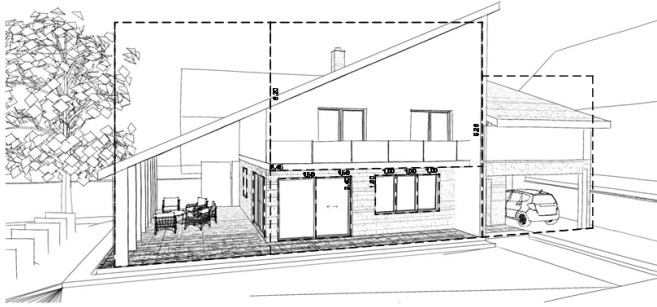


Figure 6

approximately the value of the Golden Ratio  $1,61803\dots$

Certification of aesthetic shapes is done through a statistical research (see e.g., [13, 14]).

## References

- [1] I. Crăciun and D. Inoan. On generalized Golden Ratio, *Automat. Comput. Appl. Math.*, **22** (2013), 113–119.
- [2] I. Crăciun, D. Inoan, D. Popa and L. Tudose. Generalized Golden Ratios defined by means, *Applied Mathematics and Computation*, **250** (2015), 221–227.
- [3] I. Crăciun. *Aesthetic shape optimization in industrial engineering (PhD thesis)*, UTCN, 2015.
- [4] A.R. Dunlop. *The Golden Ratio and Fibonacci Numbers*, World Scientific, 1997.
- [5] H.E. Huntley. *The Divine Proportion*, Dover Publications, New York, 1990.

- [6] M. Livio. *The Golden Ratio: The Story of Phi, the Worlds's Most Astonishing Number*, Broadway Books, 2002.
- [7] F. Rusu, I. Crăciun and L. Tudose. Aesthetic shapes generated by piecewise continuous Bézier curves, *Annals of the Tiberiu Popoviciu Seminar of Functional Equations, Approximation and Convexity*, **13** (2015), 5-16.
- [8] F. Rusu, I. Crăciun and L. Tudose. Statistical evaluation of aesthetic shapes, *Acta Technica Napocensis*, **58** (2015), 553-558.
- [9] C.P. Spears and M. Bicknell-Johnson. Asymmetric cell division: binomial identities for age analysis of mortal vs. immortal trees, *Applic. Fibonacci Numbers*, **7** (1998), 377-391.
- [10] A.P. Stakhov. The Golden Section in the measurement theory, *Computers Math. Appl.*, **17** (1998), 613-638.
- [11] A.P. Stakhov. Generalized golden sections and a new approach to geometric definition of a number, *Ukrainian Mathematical Journal*, **56**(8) (2004), 1143-1150.
- [12] A.P. Stakhov. The Generalized Principle of the Golden Section and its applications in mathematics, science, and engineering, *Chaos Solitons Fract.*, **26**(2) (2005), 263-289.
- [13] A.P. Stakhov and B. Rozin. On a new class of hyperbolic function, *Chaos Solitons Fract.*, **23**(2) (2005), 379-389.
- [14] A.P. Stakhov and B. Rozin. The "golden" algebraic equations, *Chaos Solitons Fract.*, **27**(5) (2006), 1415-1421.
- [15] A.P. Stakhov. The golden section, secrets of the Egyptian civilization and harmony mathematics, *Chaos Solitons Fract.*, **30**(2) (2006), 490-505.
- [16] A.P. Stakhov. The "golden" matrices and a new kind of cryptography, *Chaos Solitons Fract.*, **32**(3) (2007), 1138-1146.
- [17] A.P. Stakhov. *Mathematics of Harmony*, World Scientific, 2009.

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- [18] A.P. Stakhov and A. Sluchenkova. *Museum of Harmony and the Golden Section*, available from: [www.goldenmuseum.com](http://www.goldenmuseum.com), 2001.
- [19] V.D. Tsvetkov. *Heart, the Golden Section and symmetry*, Pushchino Scientific Center, 1997.