

Approximation by horseshoe kind singular integrals and applications to PDE's

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ABSTRACT. The purpose of this paper is to obtain for Lipschitz kind functions results in uniform and L^p -approximation by an interesting new real convolution singular integral, based on the so-called horseshoe probability density. The PDE satisfied by this convolution singular integral is deduced. Also, a generalized horseshoe singular integral is considered.

KEY WORDS: Horseshoe probability density, real convolution singular integral, Lipschitz functions, quantitative estimates, PDE.

MSC 2010: 41A35, 30E10, 41A25, 35C15.

1 Introduction

It is well-known the fact that the classical Gauss-Weierstrass, Poisson-Cauchy and Picard convolution singular integrals based on kernels

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suggested by the normal density function $\frac{1}{\sqrt{\pi}} \cdot e^{-x^2}$, Cauchy density function $\frac{1}{\pi} \cdot \frac{1}{1+x^2}$ and Laplace density function $\frac{1}{2} \cdot e^{-|x|}$, are given by the formulas

$$W(x, t) = \frac{1}{\sqrt{2\pi t}} \cdot \int_{-\infty}^{+\infty} f(x-u) e^{-u^2/(2t)} du, \quad C(x, t) = \frac{t}{\pi} \cdot \int_{-\infty}^{+\infty} \frac{f(x-u)}{u^2 + t^2} du,$$

$$P(x, t) = \frac{1}{2t} \cdot \int_{-\infty}^{+\infty} f(x-u) e^{-|x|/t} du, \quad x \in \mathbb{R}, t > 0,$$

respectively.

They approximate $f(x)$ as $t \searrow 0$, see, e.g., [1] and [6]. Also, it is a classical fact that $W(t, x)$ represents the solution for the heat equation $\frac{\partial W}{\partial t}(t, x) = \frac{1}{2} \frac{\partial^2 W}{\partial x^2}(t, x)$ and that $C(x, t)$ satisfies the Laplace equation $\frac{\partial^2 C}{\partial t^2}(t, x) + \frac{\partial^2 C}{\partial x^2}(t, x) = 0$, see, e.g., [8], p. 23. Recently, by using a general method based on the inverse Fourier transform technique, we proved in [7] that $P(x, t)$ represents the solution of the PDE, $\frac{\partial P}{\partial t}(t, x) = t^2 \frac{\partial^3 P}{\partial x^2 \partial t}(x, t) + 2t \frac{\partial^2 P}{\partial x^2}$ and we found the PDE's satisfied by many other convolution singular integrals, including those generated by the Maxwell-Boltzmann probability density and exponential probability density.

In [2] (see also [3]), a new interesting density function, $\frac{1}{2\pi} \cdot \ln\left(1 + \frac{1}{x^2}\right)$, called horseshoe density function is introduced and mentioned for the first time.

The aim of the present note is that by analogy with the above mentioned singular integrals, to introduce and study the approximation properties of the real convolution singular integrals based on the horseshoe density function. The PDE satisfied by this new convolution singular integral is deduced in a much simpler way than for those in [7]. Also, the generalized horseshoe singular integral is considered.

It is worth noting that while the density functions $\frac{1}{\sqrt{\pi}} \cdot e^{-x^2}$, $\frac{1}{\pi} \cdot \frac{1}{1+x^2}$ and $\frac{1}{2} \cdot e^{-|x|}$ are bounded on $\mathbb{R}$, the horseshoe density function $\frac{1}{2\pi} \cdot \ln\left(1 + \frac{1}{x^2}\right)$ is unbounded on $\mathbb{R}$ with $x = 0$ vertical asymptote (and $y = 0$ horizontal asymptote at $-\infty$ and at $+\infty$).

2 Approximation properties and PDE

The real convolution singular integral is defined by

$$\begin{aligned} H_n(f)(x) &= \frac{\sqrt{n}}{2\pi} \cdot (p.v.) \int_{-\infty}^{+\infty} f(t) \ln \left(1 + \frac{1}{n(t-x)^2} \right) dt \\ &= \frac{\sqrt{n}}{2\pi} \cdot (p.v.) \int_{-\infty}^{+\infty} f(x-v) \ln \left(1 + \frac{1}{nv^2} \right) dv, \quad n \in \mathbb{N}, \end{aligned}$$

where $f : \mathbb{R} \rightarrow \mathbb{R}$ is supposed to be such that the principal value (denoted by (p.v.)) of the integral (that is the value $V = \lim_{\varepsilon \searrow 0} \int_{-\infty}^{-\varepsilon} + \lim_{\varepsilon \searrow 0} \int_{\varepsilon}^{+\infty}$) exists finite.

For example, $H_n(f)(x)$ exists finite for any bounded f . Indeed, supposing that $|f(t)| \leq M$ for all $t \in \mathbb{R}$, then for all $x \in \mathbb{R}$ it follows

$$\begin{aligned} |H_n(f)(x)| &\leq \frac{\sqrt{n}}{2\pi} \cdot (p.v.) \int_{-\infty}^{+\infty} |f(t)| \ln \left(1 + \frac{1}{n(t-x)^2} \right) dt \\ &\leq M \cdot \frac{\sqrt{n}}{2\pi} \cdot (p.v.) \int_{-\infty}^{+\infty} \ln \left(1 + \frac{1}{n(t-x)^2} \right) dt = M. \end{aligned}$$

Indeed, taking into account that $\int \ln \left(1 + \frac{1}{x^2} \right) dx = x \ln \left(1 + \frac{1}{x^2} \right) - 2 \cot^{-1}(x) + C$ (C a constant), by the change of variable $t = \sqrt{n}(x-v)$, we easily obtain

$$\begin{aligned} \int_{\varepsilon}^{+\infty} \ln \left(1 + \frac{1}{n(x-v)^2} \right) dv &= \frac{1}{\sqrt{n}} \int_{-\infty}^{\sqrt{n}(x-\varepsilon)} \ln \left(1 + \frac{1}{t^2} \right) dt \\ &= \frac{1}{\sqrt{n}} \left[\sqrt{n}(x-\varepsilon) \ln \left(1 + \frac{1}{n(x-\varepsilon)^2} \right) - 2 \cot^{-1}(\sqrt{n}(x-\varepsilon)) + 2\pi \right]. \end{aligned}$$

Passing to limit with $\varepsilon \searrow 0$, we obtain

$$\begin{aligned} &\lim_{\varepsilon \searrow 0} \int_{\varepsilon}^{+\infty} \ln \left(1 + \frac{1}{n(x-v)^2} \right) dv \\ &= \frac{1}{\sqrt{n}} \cdot \left[x\sqrt{n} \cdot \ln \left(1 + \frac{1}{nx^2} \right) - 2 \cot^{-1}(x\sqrt{n}) + 2\pi \right]. \end{aligned}$$

By similar calculation, we obtain

$$\begin{aligned} \int_{-\infty}^{-\varepsilon} \ln \left(1 + \frac{1}{n(x-v)^2} \right) dv &= \frac{1}{\sqrt{n}} \int_{\sqrt{n}(x+\varepsilon)}^{+\infty} \ln \left(1 + \frac{1}{t^2} \right) dt \\ &= \frac{1}{\sqrt{n}} \left[-\sqrt{n}(x+\varepsilon) \ln \left(1 + \frac{1}{n(x+\varepsilon)^2} \right) + 2 \cot^{-1}(\sqrt{n}(x+\varepsilon)) \right]. \end{aligned}$$

Passing to limit with $\varepsilon \searrow 0$, we obtain

$$\begin{aligned} \lim_{\varepsilon \searrow 0} \int_{-\infty}^{-\varepsilon} \ln \left(1 + \frac{1}{n(x-v)^2} \right) dv \\ = \frac{1}{\sqrt{n}} \cdot \left[-x\sqrt{n} \cdot \ln \left(1 + \frac{1}{nx^2} \right) + 2 \cot^{-1}(x\sqrt{n}) \right]. \end{aligned}$$

Therefore, it follows

$$(p.v.) \int_{-\infty}^{+\infty} \ln \left(1 + \frac{1}{n(t-x)^2} \right) dt = \frac{2\pi}{\sqrt{n}},$$

which finally leads to the required inequality.

The first main result of this section is the following.

Theorem 2.1. *Let $\alpha \in (0, 1)$.*

(i) *Let us suppose that f is bounded and belongs to the class denoted by $Lip \alpha$, consisting in the class of all functions $f : \mathbb{R} \rightarrow \mathbb{R}$ such that there exists a constant $C > 0$ (depending on f), with the property that $|f(x) - f(y)| \leq C|x - y|^\alpha$, for all $x, y \in \mathbb{R}$. For all $x \in \mathbb{R}$ and $n \in \mathbb{N}$ we have*

$$|H_n(f)(x) - f(x)| \leq C_{\alpha, f} \cdot \frac{1}{n^{\alpha/2}},$$

where $C_{\alpha, f} > 0$ depends on α and f but is independent of n .

(ii) *Let $1 \leq p < +\infty$ and suppose that $f \in L^p(\mathbb{R}) = \{f : \mathbb{R} \rightarrow \mathbb{R}; \int_{\mathbb{R}} |f(x)|^p dx < +\infty\}$ is such that*

$$\int_{-\infty}^{+\infty} |f(x+h) - f(x)|^p dx \leq c_{p, f} \cdot |h|^\alpha, \text{ for all } h \in \mathbb{R}.$$

Then, for all $n \in \mathbb{N}$ we have

$$\|H_n(f) - f\|_p^p \leq C_{f,\alpha,p} \cdot \frac{1}{n^{\alpha/2}},$$

where $\|f\|_p = (\int_{\mathbb{R}} |f(x)|^p dx)^{1/p}$ and $C_{f,\alpha,p} > 0$ depends on f, α, p but is independent of n .

Proof. (i) Since by the calculation made just before the statement of Theorem 2.1 we have $H_n(1)(x) = 1$, it immediately follows

$$\begin{aligned} |H_n(f)(x) - f(x)| &= \left| \frac{\sqrt{n}}{2\pi} \cdot (p.v.) \int_{-\infty}^{+\infty} (f(x-v) - f(x)) \ln \left(1 + \frac{1}{nv^2} \right) dv \right| \\ &\leq \frac{\sqrt{n}}{2\pi} \cdot (p.v.) \int_{-\infty}^{+\infty} |f(x-v) - f(x)| \ln \left(1 + \frac{1}{nv^2} \right) dv \\ &\leq \frac{C \cdot \sqrt{n}}{2\pi} \cdot (p.v.) \int_{-\infty}^{+\infty} |v|^\alpha \ln \left(1 + \frac{1}{nv^2} \right) dv \\ &= \frac{C \cdot \sqrt{n}}{\pi} \cdot (p.v.) \int_0^{+\infty} v^\alpha \ln \left(1 + \frac{1}{nv^2} \right) dv \\ &= \frac{C}{\pi} \cdot \frac{1}{n^{\alpha/2}} \cdot (p.v.) \int_0^{+\infty} t^\alpha \ln \left(1 + \frac{1}{t^2} \right) dt. \end{aligned}$$

But integrating by parts and then applying the L'Hopital's rule, we easily get

$$(p.v.) \int_0^{+\infty} t^\alpha \ln \left(1 + \frac{1}{t^2} \right) dt = \frac{2}{\alpha + 1} \cdot \int_0^{+\infty} \frac{t^\alpha}{1 + t^2} dt,$$

where it is easy to show that for $0 < \alpha < 1$ we have $\int_0^{+\infty} \frac{t^\alpha}{1+t^2} dt < +\infty$.

This implies the required estimate.

(ii) We will use the following well-known Jensen type inequality for integrals : if $a, b \in \mathbb{R}$, $a < b$ and if $\int_a^b G(u) du = 1$, $G(u) \geq 0$ for all $u \in (a, b)$ and $\varphi(t)$ is convex function over the range of the measurable function of real variable F , then

$$\varphi \left(\int_a^b F(u) G(u) du \right) \leq \int_a^b \varphi(F(u)) G(u) du.$$

Also, the inequalities $(a + b)^p \leq 2^{p-1}(a^p + b^p) \leq 2^p(a + b)^p$ valid for all $a, b \geq 0, 1 \leq p < +\infty$, will be useful.

We get

$$|H_n(f)(x) - f(x)| \leq \frac{\sqrt{n}}{2\pi} \cdot (p.v.) \int_{-\infty}^{+\infty} |f(x - v) - f(x)| \ln \left(1 + \frac{1}{nv^2} \right) dv,$$

which implies

$$\begin{aligned} & |H_n(f)(x) - f(x)|^p \\ & \leq \left(\frac{\sqrt{n}}{2\pi} \cdot (p.v.) \int_{-\infty}^{+\infty} |f(x - v) - f(x)| \ln \left(1 + \frac{1}{nv^2} \right) dv \right)^p \\ & = \lim_{\varepsilon \searrow 0} \left[\frac{\sqrt{n}}{2\pi} \int_{-\infty}^{-\varepsilon} |f(x - v) - f(x)| \ln \left(1 + \frac{1}{nv^2} \right) dv \right. \\ & \quad \left. + \frac{\sqrt{n}}{2\pi} \int_{\varepsilon}^{+\infty} |f(x - v) - f(x)| \ln \left(1 + \frac{1}{nv^2} \right) dv \right]^p \\ & \leq 2^{p-1} \lim_{\varepsilon \searrow 0} \left(\frac{\sqrt{n}}{2\pi} \int_{-\infty}^{-\varepsilon} |f(x - v) - f(x)| \ln \left(1 + \frac{1}{nv^2} \right) dv \right)^p \\ & \quad + 2^{p-1} \lim_{\varepsilon \searrow 0} \left(\frac{\sqrt{n}}{2\pi} \int_{\varepsilon}^{+\infty} |f(x - v) - f(x)| \ln \left(1 + \frac{1}{nv^2} \right) dv \right)^p. \end{aligned}$$

Now, taking into account the calculations in this section made just before the statement of Theorem 2.1 and denoting

$$\begin{aligned} 0 & < A_{n,x,\varepsilon} := \frac{\sqrt{n}}{2\pi} \int_{-\infty}^{-\varepsilon} \ln \left(1 + \frac{1}{n(x - v)^2} \right) dv \\ & = \frac{1}{2\pi} \left[-\sqrt{n}(x + \varepsilon) \ln \left(1 + \frac{1}{n(x + \varepsilon)^2} \right) + 2 \cot^{-1}(\sqrt{n}(x + \varepsilon)) \right] < +\infty \end{aligned}$$

and

$$\begin{aligned} B_{n,x,\varepsilon} & := \frac{\sqrt{n}}{2\pi} \int_{\varepsilon}^{+\infty} \ln \left(1 + \frac{1}{n(x - v)^2} \right) dv \\ & = \frac{1}{2\pi} \left[\sqrt{n}(x - \varepsilon) \ln \left(1 + \frac{1}{n(x - \varepsilon)^2} \right) - 2 \cot^{-1}(\sqrt{n}(x - \varepsilon)) + 2\pi \right], \end{aligned}$$

it follows

$$\int_{-\infty}^{-\varepsilon} \left[\frac{\sqrt{n}}{2\pi} \cdot \frac{1}{A_{n,x,\varepsilon}} \ln \left(1 + \frac{1}{n(x-v)^2} \right) \right] dv = 1$$

and

$$\int_{\varepsilon}^{+\infty} \left[\frac{\sqrt{n}}{2\pi} \cdot \frac{1}{B_{n,x,\varepsilon}} \ln \left(1 + \frac{1}{n(x-v)^2} \right) \right] dv = 1.$$

Replacing in the above inequality and applying the Jensen's inequality for $\varphi(t) = t^p$, we obtain

$$\begin{aligned} & |H_n(f)(x) - f(x)|^p \\ & \leq 2^{p-1} \lim_{\varepsilon \searrow 0} \left(A_{n,x,\varepsilon} \cdot \int_{-\infty}^{-\varepsilon} |f(x-v) - f(x)| \left[\frac{\sqrt{n}}{2\pi} \cdot \frac{1}{A_{n,x,\varepsilon}} \ln \left(1 + \frac{1}{nv^2} \right) \right] dv \right)^p \\ & \quad + 2^{p-1} \lim_{\varepsilon \searrow 0} \left(B_{n,x,\varepsilon} \cdot \int_{\varepsilon}^{+\infty} |f(x-v) - f(x)| \left[\frac{\sqrt{n}}{2\pi} \cdot \frac{1}{B_{n,x,\varepsilon}} \ln \left(1 + \frac{1}{nv^2} \right) \right] dv \right)^p \\ & \leq 2^{p-1} \lim_{\varepsilon \searrow 0} \left(A_{n,x,\varepsilon} \cdot \int_{-\infty}^{-\varepsilon} |f(x-v) - f(x)|^p \left[\frac{\sqrt{n}}{2\pi} \cdot \frac{1}{A_{n,x,\varepsilon}} \ln \left(1 + \frac{1}{nv^2} \right) \right] dv \right) \\ & \quad + 2^{p-1} \lim_{\varepsilon \searrow 0} \left(B_{n,x,\varepsilon} \cdot \int_{\varepsilon}^{+\infty} |f(x-v) - f(x)|^p \left[\frac{\sqrt{n}}{2\pi} \cdot \frac{1}{B_{n,x,\varepsilon}} \ln \left(1 + \frac{1}{nv^2} \right) \right] dv \right). \end{aligned}$$

Simplifying inside of the parenthesis with $A_{n,x,\varepsilon}$ and $B_{n,x,\varepsilon}$, integrating the above inequality with respect to $x \in \mathbb{R}$ and applying the Fubini's theorem, we easily obtain

$$\begin{aligned} & \|H_n(f) - f\|_p^p \\ & \leq 2^{p-1} \cdot (p.v.) \int_{-\infty}^{+\infty} \left[\int_{-\infty}^{+\infty} |f(x-v) - f(x)|^p dx \right] \frac{\sqrt{n}}{2\pi} \ln \left(1 + \frac{1}{nv^2} \right) dv \\ & \leq c_{f,p} \cdot 2^{p-1} (p.v.) \int_{-\infty}^{+\infty} |v|^\alpha \frac{\sqrt{n}}{2\pi} \ln \left(1 + \frac{1}{nv^2} \right) dv \\ & = c_{f,p} \cdot 2^p (p.v.) \int_0^{+\infty} v^\alpha \frac{\sqrt{n}}{2\pi} \ln \left(1 + \frac{1}{nv^2} \right) dv \\ & \leq 2^p \cdot C_{f,p,\alpha} \cdot \frac{1}{n^{\alpha/2}}, \end{aligned}$$

which implies the estimate in the statement. $\square$

Remark 2.2. If $\alpha = 1$, then the reasonings for the statement of Theorem 2.1, (i), do not work. Indeed, in this case we get

$$(p.v.) \int_0^{+\infty} v \ln \left(1 + \frac{1}{nv^2} \right) dv = \frac{1}{n} \cdot (p.v.) \int_0^{+\infty} t \ln \left(1 + \frac{1}{t^2} \right) dt$$

and

$$\int t \ln \left(1 + \frac{1}{t^2} \right) dt = \frac{1}{2} \cdot x^2 \cdot \ln \left(1 + \frac{1}{t^2} \right) + \frac{1}{2} \cdot \ln(1 + x^2) + C,$$

where C is a real constant. Then, it follows

$$\int_0^{+\infty} t \ln \left(1 + \frac{1}{t^2} \right) dt = \left[\frac{1}{2} \cdot x^2 \cdot \ln \left(1 + \frac{1}{x^2} \right) + \frac{1}{2} \cdot \ln(1 + x^2) \right]_0^{+\infty} = +\infty.$$

In what follows, it is natural to ask what partial differential equation would satisfy $H_t(f)(x) = (p.v.) \int_{-\infty}^{+\infty} f(x-v) \cdot d(t,v)dv$, where $d(t,x) = \frac{1}{2t\pi} \ln \left(1 + \frac{t^2}{x^2} \right)$.

Here, by Theorem 2.1, (i), we have that $\lim_{t \searrow 0} H_t(f)(x) = f(x)$ for all $x \in \mathbb{R}$ and for f Lipschitz α function with $0 < \alpha < 1$.

In this sense, we have :

Theorem 2.3. *Let f be bounded, of Lipschitz class $\alpha \in (0,1)$ on $\mathbb{R}$, with f' bounded and uniformly continuous on $\mathbb{R}$. Then $u(x,t) = H_t(f)(x)$ is solution of the initial value problem*

$$\frac{\partial u}{\partial t}(x,t) = \frac{\partial u}{\partial x}(x,t) - \frac{1}{t}u(x,t), \quad \lim_{s \searrow 0} u(x,s) = f(x), t > 0, x \in \mathbb{R}.$$

Proof. Denoting $u(x,t) = \frac{1}{2\pi t} (p.v.) \int_{-\infty}^{+\infty} f(x-v) \cdot \ln \left(1 + \frac{t^2}{v^2} \right) dv$, we

formally get

$$\begin{aligned}
\frac{\partial u}{\partial t}(x, t) &= \frac{1}{2\pi} (p.v.) \int_{-\infty}^{+\infty} f(x-v) \left(\frac{\ln(1+t^2/v^2)}{t} \right)'_t dv \\
&= \frac{1}{2\pi t} (p.v.) \int_{-\infty}^{+\infty} f(x-v) \cdot \frac{2t}{v^2+t^2} dv \\
&\quad - \frac{1}{t} \cdot \frac{1}{2\pi t} (p.v.) \int_{-\infty}^{+\infty} f(x-v) \ln(1+t^2/v^2) dv \\
&= \frac{1}{2\pi t} (p.v.) \int_{-\infty}^{+\infty} f(x-v) \cdot d[\ln(1+t^2/v^2)] - \frac{1}{t} u(x, t) \\
&= \frac{1}{2\pi t} \cdot f(x-v) \cdot \ln(1+t^2/v^2) \Big|_{-\infty}^{+\infty} \\
&\quad + \frac{1}{2\pi t} (p.v.) \int_{-\infty}^{+\infty} f'(x-v) \ln(1+t^2/v^2) dv - \frac{1}{t} u(x, t) \\
&= \frac{1}{2\pi t} \cdot f(x-v) \cdot \ln(1+t^2/v^2) \Big|_{-\infty}^{+\infty} + \frac{\partial u}{\partial x}(x, t) - \frac{1}{t} u(x, t).
\end{aligned}$$

Since by Theorem 2.1, (i), f is uniformly continuous on $\mathbb{R}$, it follows that (see, e.g., [4], p. 48, Problème 4 or [5]) f is of linear growth, that is $|f(x)| \leq \alpha|x| + \beta$, for all $x \in \mathbb{R}$, with $\alpha, \beta \geq 0$. This immediately implies that $\frac{1}{2\pi t} \cdot f(x-v) \cdot \ln(1+t^2/v^2) \Big|_{-\infty}^{+\infty} = 0$, which leads to the fact that $u(x, t) = H_t(f)(x)$ is solution of the initial value problem in the statement. $\square$

Remark 2.4. We can consider the generalized horseshoe singular integral, which for any $n \in \mathbb{N}$ can be defined by

$$\begin{aligned}
&H_{n,t}(f)(x) \\
&= -\frac{1}{2t\pi} (p.v.) \int_{-\infty}^{+\infty} \left(\sum_{k=1}^{n+1} (-1)^k \binom{n+1}{k} f(x+kv) \ln \left(1 + \frac{t^2}{v^2} \right) \right) dv \\
&= (p.v.) \int_{-\infty}^{+\infty} f(x-u) \left[\sum_{k=1}^{n+1} (-1)^{k+1} \binom{n+1}{k} \cdot \frac{1}{k} \cdot \frac{\ln \left(1 + \frac{k^2 t^2}{u^2} \right)}{2t\pi} \right] du \\
&= \sum_{k=1}^{n+1} (-1)^{k+1} \binom{n+1}{k} H_{kt}(f)(x),
\end{aligned}$$

where $H_{kt}(f)(x) = \frac{1}{2\pi kt}(p.v.) \int_{-\inf ty}^{+\infty} f(x-u) \ln \left(1 + \frac{k^2 t^2}{u^2}\right) du$.

Theorem 2.5. *Let f be bounded, of Lipschitz class $\alpha \in (0, 1)$ on $\mathbb{R}$, with f' bounded and uniformly continuous on $\mathbb{R}$. Then, for any $n \in \mathbb{N}$, $u(x, t) := H_{n,t}(f)(x)$ also is solution of the initial value problem*

$$\frac{\partial u}{\partial t}(x, t) = \frac{\partial u}{\partial x}(x, t) - \frac{1}{t}u(x, t), \quad \lim_{s \searrow 0} u(x, s) = f(x), \quad t > 0, x \in \mathbb{R}.$$

Proof. We have

$$H_{n,t}(f)(x) = \sum_{k=1}^{n+1} (-1)^{k+1} \binom{n+1}{k} u_k(x, t).$$

We firstly prove that each

$$u_k(x, t) = H_{kt}(f)(x) = \frac{1}{2\pi kt}(p.v.) \int_{-\infty}^{+\infty} f(x-v) \ln \left(1 + \frac{k^2 t^2}{v^2}\right) dv,$$

$k = 1, \dots, n+1$, is solution of the initial value problem in the statement. For this purpose, we reason exactly as in the proof of Theorem 2.3.

Indeed, similar calculation with those in the proof of Theorem 2.3, lead to

$$\begin{aligned} \frac{\partial u_k}{\partial t}(x, t) &= \frac{1}{2\pi kt} \cdot f(x-v) \cdot \ln(1 + k^2 t^2 / v^2) \Big|_{-\infty}^{+\infty} + \frac{\partial u_k}{\partial x}(x, t) - \frac{1}{t}u_k(x, t) \\ &= \frac{\partial u_k}{\partial x}(x, t) - \frac{1}{t}u_k(x, t). \end{aligned}$$

Also, from Theorem 2.1, (i), it follows that $\lim_{t \searrow 0} u_k(x, t) = f(x)$, for all $x \in \mathbb{R}$.

The proof is terminated by taking into account that $H_{n,t}$ is a linear combination of u_k and $\sum_{k=1}^{n+1} (-1)^{k+1} \binom{n+1}{k} = 1$. $\square$

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