

On the Bézier cubic of best approximation to a plane curve

VICUȚA NEAGOS
(CLUJ-NAPOCA)

ABSTRACT. We obtain some numerical estimates for the Bézier cubic of the best approximation to a circle and other plane curves.

KEY WORDS: Bézier cubics, Hausdorff metric

MSC 2010: 41A05, 41A10, 65D17.

1 Introduction

Responding to the request from the automobile industry to develop a modern design system for the curved shape of cars, *Paul de Faget de CASTELJAU* (b. 1930) at Citroën (1959), and *Pierre Etienne BÉZIER* (1910–1999), at Renault (1962), had developed what is now known as the theory of *Bézier curves and surfaces*.

Because de Casteljau never published his results, many of the notions in this theory (and of course, the theory itself) are named after Bézier.

Let $p, n \geq 2$ be integers, and $(M_0, \dots, M_n)$ an ordered system of points from $\mathbb{R}^p$.

[◇]Vicuța Neagos, Technical University of Cluj-Napoca
email: vicuta.neagos@math.utcluj.ro

Definition 1.1 *The curve defined by*

$$B[M_0, \dots, M_n](t) := \sum_{k=0}^n \binom{n}{k} (1-t)^{n-k} t^k M_k, \quad t \in [0, 1],$$

is called the Bézier curve related to the system $(M_0, \dots, M_n)$.

The points M_i are called *control points* or *Bézier points*. The curve $B[M_0, \dots, M_n]$ *mimes* the form of the polygon $[M_0 \dots M_n]$ called the *Bézier polygon* or *control polygon* and has the following essential properties in Computer Aided Design:

- it begins at M_0 , heading in the direction $\overline{M_0 M_1}$;
- it ends at M_n , heading in the direction $\overline{M_{n-1} M_n}$;
- it stays entirely within the convex hull of the set $\{M_0, \dots, M_n\}$.

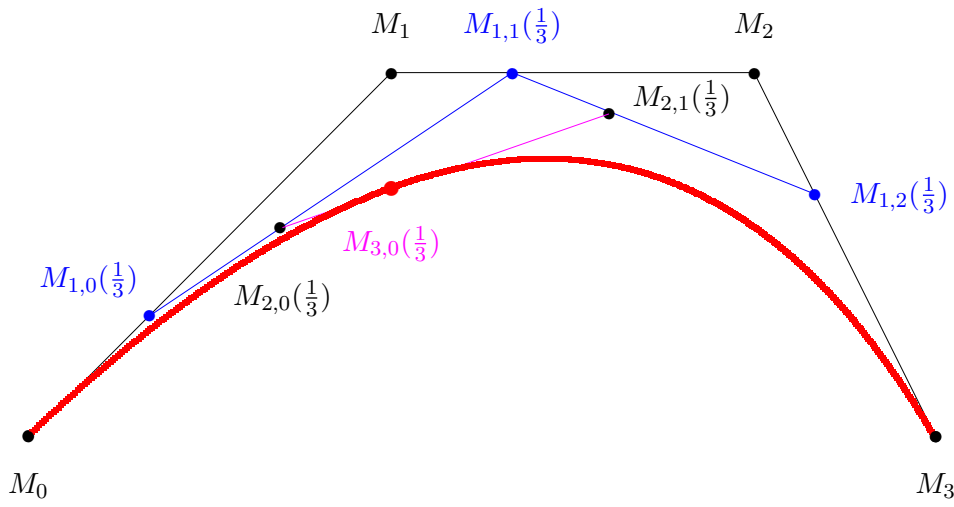
A fundamental recursive method used to determine the curve $B[M_0, \dots, M_n]$ is the *Popoviciu-Casteljau Algorithm* presented by Tiberiu Popoviciu [12, p. 39, Eq. (29)] at classes since 1933 (many years before Casteljau [6]), and described below.

Let

$$\begin{aligned} M_{0,i}(t) &= M_i, \\ M_{r,i}(t) &= (1-t)M_{r-1,i}(t) + tM_{r-1,i+1}(t) \\ &\quad (r = 1, \dots, n; i = 0, \dots, n-r). \end{aligned}$$

The $M_{n,0}(t)$ is the point with parameter value t on the Bézier curve $B[M_0, \dots, M_n]$.

An example for $n = 3$ and $t = 1/3$ is given below.



On the fence around the tomb of Tiberiu Popoviciu (1906-1975) appears, in wrought iron, a sketch of the Popoviciu-Casteljau algorithm.

2 The Metafont system

The characters used in typesetting are known as *fonts* or *type*.

Albrecht Dürer (1471–1528) and other Renaissance men attempted to establish mathematical principles of font design, but the letters they obtained did not correspond to the aesthetic requirements. Their method failed because they restricted themselves to “ruler and compass” constructions, which cannot adequately express the nuances of good calligraphy [10, p. 13]. METAFONT [10], a computer system for the design of fonts, solves this problem using cubic Bézier curves.

In this context, about its author, Professor Donald Ervin Knuth, it is said that “he took the lead out of print”.

The basic idea is to use sets of four control points M_0, M_1, M_2, M_3 , so arranged that the Bézier curves $B_3[M_0, M_1, M_2, M_3]$ are connected and define the border of the plane set points corresponding to a character.

The curves $B_3[M_0, M_1, M_2, M_3]$, having as components Bernstein polynomials of degree 3, are often called *Bézier cubics*.

The method of Bézier cubics is good enough to obtain a satisfactory approximation to any curve. A more difficult problem is finding the sets of four control points. To obtain a smooth curve, two adjacent cubics must be of the form $B_3[M_0, M_1, M_2, M_3]$, $B_3[M_3, M_4, M_5, M_6]$, where M_2, M_3, M_4 are collinear.

3 The Hausdorff distance and the radial distance of two curves

In this section, we establish sufficient conditions for the equality of the Hausdorff distance and the radial distance between a circle and a plane curve.

Let $\mathbf{C}$ be the quarter of the unit circle in the first quadrant

$$(3.1) \quad \mathbf{C}(t) = (\cos t, \sin t), \quad t \in \left[0, \frac{\pi}{2}\right]$$

and Γ be a curve also in the first quadrant parametrized by

$$\Gamma(s) = (p(s), q(s)), \quad s \in [0, 1],$$

with $p, q : [0, 1] \rightarrow [0, \infty)$ continuous.

With $r(s) := \sqrt{p^2(s) + q^2(s)}$, the *radial distance* between $\mathbf{C}$ and Γ , is defined by

$$(3.2) \quad d_{\text{rad}}(\mathbf{C}, \Gamma) = \max_s |r(s) - 1|.$$

In general, if $r = r_1(\theta)$, $r = r_2(\theta)$, with θ in an interval I , are the polar representations of two curves Γ_1 and Γ_2 , then we define the radial distance between the curves Γ_1 and Γ_2 by

$$d_{\text{rad}}(\Gamma_1, \Gamma_2) := \max_{\theta \in I} |r_1(\theta) - r_2(\theta)|.$$

Assume that

$$\Gamma(0) = \mathbf{C}(0), \quad \Gamma(1) = \mathbf{C}\left(\frac{\pi}{2}\right),$$

i.e., $p(1) = q(0) = 0$ and $p(0) = q(1) = 1$.

Let ρ be the usual Euclidian distance in the plane and recall the Hausdorff distance between $\mathbf{C}$ and Γ ,

$$d_{\text{H}}(\mathbf{C}, \Gamma) := \max \left\{ \max_s \min_t f(s, t), \max_t \min_s f(s, t) \right\},$$

with $f(s, t) = \rho(\mathbf{C}(t), \Gamma(s))$.

Theorem 3.1 *Under the previous assumptions, the radial distance $d_{\text{rad}}(\mathbf{C}, \Gamma)$, and the Hausdorff distance $d_{\text{H}}(\mathbf{C}, \Gamma)$ are equal to each other.*

Proof. Define the function $\theta : [0, 1] \rightarrow \left[0, \frac{\pi}{2}\right]$,

$$\theta(s) = \begin{cases} \arctan \frac{q(s)}{p(s)}, & p(s) \neq 0, \\ \frac{\pi}{2}, & p(s) = 0. \end{cases}$$

Note that if $p^2(s) + q^2(s) > 0$ for all $s \in [0, 1]$, then θ is continuous and onto. We have an alternative representation of $\mathbf{\Gamma}$,

$$\mathbf{\Gamma}(s) = (r(s) \cos \theta(s), r(s) \sin \theta(s)), \quad s \in [0, 1],$$

which leads to

$$\begin{aligned} f(s, t) &= \sqrt{(p(s) - \cos t)^2 + (q(s) - \sin t)^2} \\ &= \sqrt{r^2(s) - 2r(s) \cos(\theta(s) - t) + 1}, \quad s \in [0, 1], \quad t \in \left[0, \frac{\pi}{2}\right]. \end{aligned}$$

Since $|\theta(s) - t| \leq \frac{\pi}{2}$, it follows that

$$\min_t f(s, t) = f(s, \theta(s)) = |r(s) - 1| \quad \text{for all } s \in [0, 1],$$

hence

$$\max_s \min_t f(s, t) = d_{\text{rad}}(\mathbf{C}, \mathbf{\Gamma}).$$

Next, we consider two cases:

Case 1: $r(s) \neq 0$, for all $s \in [0, 1]$.

Let $t \in \left[0, \frac{\pi}{2}\right]$. Since θ is onto, there exists $s_t \in [0, 1]$ such that $\theta(s_t) = t$. Then

$$\min_s f(s, t) \leq f(s_t, t) = f(s_t, \theta(s_t)) = |r(s_t) - 1|$$

hence

$$\max_t \min_s f(s, t) \leq \max_t |r(s_t) - 1| \leq \max_s |r(s) - 1| = d_{\text{rad}}(\mathbf{C}, \mathbf{\Gamma}).$$

Case 2: there exists $s_0 \in [0, 1]$ such that $r(s_0) = 0$.

For all $t \in \left[0, \frac{\pi}{2}\right]$, we have:

$$\min_s f(s, t) \leq f(s_0, t) = 1 = |r(s_0) - 1| \leq d_{\text{rad}}(\mathbf{C}, \mathbf{\Gamma}),$$

hence

$$\max_t \min_s f(s, t) \leq d_{\text{rad}}(\mathbf{C}, \Gamma).$$

Concluding,

$$d_H(\mathbf{C}, \Gamma) = \max \left\{ \max_s \min_t f(s, t), \max_t \min_s f(s, t) \right\} = d_{\text{rad}}(\mathbf{C}, \Gamma).$$

□

Remark 3.1 *Under the assumptions of Theorem 3.1, the Hausdorff distance and the radial distance are equal to each other (see also [9] and the references therein), but, in general, this is not true.*

Indeed, let us consider the sets

$$\mathcal{C}_1 := \{(x, y) \in \mathbb{R}^2 \mid (x-1)^2 + (y-1)^2 = 1, x+y \leq 1\},$$

$$\mathcal{C}_2 := \{(x, y) \in \mathbb{R}^2 \mid (x-1)^2 + (y-1)^2 = 1, x+y \geq 1\}.$$

The radial distance between $\mathcal{C}_1$ and $\mathcal{C}_2$ is 2, while the Hausdorff distance is $\sqrt{2 + \sqrt{2}}$.

4 Approximation of circle by cubic Bézier curves

Let $a \in [0, 1]$. We consider the system of points

$$(P_1 = (0, 1), P_2 = (a, 1), P_3 = (1, a), P_4 = (1, 0)),$$

and the associated Bézier cubic curve,

$$(4.1) \quad \mathbf{B}_a(t) := (1-t)^3 P_1 + 3t(1-t)^2 P_2 + 3t^2(1-t) P_3 + t^3 P_4,$$

where,

$$\mathbf{B}_a(t) = (p_a(t), q_a(t)),$$

with

$$\begin{aligned} p_a(t) &:= 3a(1-t)^2 t + 3(1-t)t^2 + t^3, \\ q_a(t) &:= 3at^2(1-t) + 3t(1-t)^2 + (1-t)^3, \quad t \in [0, 1]. \end{aligned}$$

A problem raised since 1986 is to find a control polygon such that the

associated Bézier cubic approximates the circle as best as possible, more precisely, find $a_* \in [0, 1]$ such that $\mathbf{B}_{a_*}$ be the best approximation to the quarter of the unit circle $\mathbf{C}$ (3.1) by a Bézier cubic (4.1), in sense of the radial distance $d_{\text{rad}}(\mathbf{B}_a, \mathbf{C}) := \max_{t \in [0, 1]} \left| \sqrt{p_a^2(t) + q_a^2(t)} - 1 \right|$, i.e.,

$$d_{\text{rad}}(\mathbf{B}_{a_*}, \mathbf{C}) = \min_{a \in [0, 1]} d_{\text{rad}}(\mathbf{B}_a, \mathbf{C}).$$

Many authors have made substantial contributions to solving this problem: [7], [8], [4], [15], [1], [3], [11], [14], [2], [13], [5].

Using direct techniques from classical analysis to find the extrema, Spencer Mortensen [11] proved that the value of the parameter a which minimizes the maximum radial distance from the circle $\mathbf{C}$ to the Bézier curve $\mathbf{B}_a$ is $a_* = 0.55191502449\dots$.

The solution of Mortensen is ingenious and brief. We reveal below the hidden steps of his proof.

Let $F(a, t) := p_a^2(t) + q_a^2(t)$, hence

$$\begin{aligned} F(a, t) = & 1 + 3(3a^2 + 2a - 2)t^2 - 4(9a^2 - 3a - 1)t^3 + 3(21a^2 - 22a + 6)t^4 \\ & - 6(3a - 2)^2 t^5 + 2(3a - 2)^2 t^6. \end{aligned}$$

We have

$$\frac{\partial F(a, t)}{\partial t} = 6(t - 1)t(2t - 1) \left((3a - 2)^2 t^2 - (3a - 2)^2 t + 3a^2 + 2a - 2 \right).$$

Reducing the conditions $-3a^2 - 20a + 12 > 0 \wedge a \neq \frac{2}{3} \wedge 0 \leq a \leq 1$ gives $0 \leq a < \frac{1}{3}(2\sqrt{34} - 10) = 0.553968\dots$, and the roots of the derivative $\frac{\partial F(a, t)}{\partial t}$ are:

$$0, \quad \frac{1}{2}, \quad 1, \quad \frac{1}{2} \pm \frac{\sqrt{12 - 20a - 3a^2}}{4 - 6a}.$$

This radial distance function, $dR(a, t) := \left| \sqrt{p_a^2(t) + q_a^2(t)} - 1 \right|$, has maxima at $t = \frac{1}{2}$ and $t = \frac{1}{2} \pm \frac{\sqrt{12 - 20a - 3a^2}}{4 - 6a}$. Mortensen imposed the condition

that all maxima have the same value. Because the Bézier curve is symmetric about $t = \frac{1}{2}$, for $0 \leq a < \frac{1}{3}(2\sqrt{34} - 10)$, this condition is equivalent to the following equality

$$(4.2) \quad \text{dR}\left(a, \frac{1}{2}\right) = \text{dR}\left(a, \frac{1}{2} + \frac{\sqrt{12 - 20a - 3a^2}}{4 - 6a}\right).$$

After laborious calculations, we deduce that the roots of (4.2) are among the roots of the polynomial

$$(4.3) \quad \begin{aligned} P(a) &= (a - 0)(a - 8/15)(a + 2/3(5 + \sqrt{34}))^2(a + 2/3(5 - \sqrt{34}))^2 \\ &\quad \times \left(552960\sqrt{2} - 1105920 + (6013440 - 2903040\sqrt{2})a \right. \\ &\quad \left. - (11689920 - 4976640\sqrt{2})a^2 + (9624960 - 1866240\sqrt{2})a^3 \right. \\ &\quad \left. - (3460320 + 2799360\sqrt{2})a^4 + (933120 + 2099520\sqrt{2})a^5 + 72900a^6 \right). \end{aligned}$$

The MATHEMATICA software program suggests that the equation $P = 0$ is not solvable by radicals. Using the MATHEMATICA command

Solve[P[a] == 0, a, PositiveReals]

we obtain the positive roots of P and the value of $\text{dR}(\cdot, 1/2)$ on these roots:

a	$\frac{8}{15}$	$\frac{2}{3}(\sqrt{34} - 5)$	$0.551915\dots$
$\text{dR}(a, 1/2)$	0.0099498	0.000892281	0.000172333

From the previous table, we deduce that the best approximation Bézier cubic to the quarter of circle (3.1) is obtained for $a = 0.551915\dots$

Since we still cannot get a sufficiently accurate value of the parameter a without MATHEMATICA, we aim to use *only* MATHEMATICA:

```
(* A Short MATHEMATICA Approach *)
points[a_]:={{0,1},{a,1},{1,a},{1,0}}
Bezier[a_,t_]:=Sum[Binomial[3,i]t^i(1-t)^(3-i)points[a][[i+1]],{i,0,3}]
(* We avoid using "Abs" in "Minimize" and in "Maximize" *)
dR[a_,t_]:=Sqrt[(Bezier[a,t][[1]]^2+(Bezier[a,t][[2]]^2-1)^2]
Minimize[{First[Maximize[{dR[a,t],0<=t<=1},t]],0<=a<=1},a]/N
{0.000196076,{a->0.551915}}
```

Let us now compare the result of Mortensen with other approaches.

Imposing interpolation conditions on the Bézier cubic, in [7, Theorem 1] the authors obtained (for $\alpha = \pi/2$)

$$a_0 = \frac{4}{3} \tan\left(\frac{\pi}{8}\right).$$

From some geometric conditions on the Bézier polygon, in [15, Eq. (5)], the author obtained the value

$$a_1 = \frac{4}{3} (\sqrt{2} - 1).$$

We realize just now that the value obtained by Riskus (2006) is nothing but that obtained by Dokken et al. (1990).

Indeed we have

$$a_0 = \frac{4}{3} \tan\left(\frac{\pi}{8}\right) = \frac{4}{3} (\sqrt{2} - 1) = a_1 = 0.5522847498 \dots$$

We calculated

$$d_{\text{rad}}(\mathbf{B}_{a_0}, \mathbf{C}) \approx 0.00027 \dots,$$

which is a slightly worse alternative than that of Mortensen.

“Radial deviation” is not conducive to the measurement of the “distance” between two characters, which are sets of points in the plane. Therefore, we use the Hausdorff-Pompeiu distance.

Let $\mathcal{F}$ be the family of all bounded and closed sets in a metric space (X, ρ) . The *Hausdorff-Pompeiu* distance on $\mathcal{F}$ is defined by

$$(4.4) \quad d_H(A, B) := \max \left\{ \sup_{a \in A} \inf_{b \in B} \rho(a, b), \sup_{b \in B} \inf_{a \in A} \rho(a, b) \right\}$$

We propose to find an approximation of the parameter a , which minimizes the Hausdorff distance

$$d_H(\mathbf{B}_a, \mathbf{C}).$$

5 The MATHEMATICA code for finding the best approximation Bézier cubic of a circle in Hausdorff metric

The definition formula of the Hausdorff metric (4.4) prevented us from finding an analytic solution (implies solving some systems of transcendental equations with several unknowns).

Note that even in the particular case where the approximate curve is a circle and the distance is the radial one, we did not manage to find a numerical approximation without the help of a mathematical assistance software.

Even more challenging is finding a solution in case of the Hausdorff distance and more complicated curves.

Thus, we estimate the element of the best approximation numerically in the Hausdorff metric, by a MATHEMATICA code. In this code, we avoid using the predefined algorithms as “FindMin,” “FindMax” (which return local extrema or fail in finding them out), “Minimize” and “Maximize” (attempt to find a global extremum numerically, but may find a local extremum). We only use “secure” commands (for example, “Min” and “Max”), for approximating the solution to the problem. However, we do not deny that more efficient algorithms could be provided.

We emphasize that because we use only the comparison as the main instruction, the algorithm does not cause the propagation of arithmetic errors. Note that if the functions are continuous on $[0, 1] \times [0, 1]$ (so uniformly

continuous), taking a partition of the interval $[0, 1]$ fine enough ensures the desired precision of the estimate of extrema.

However, being a brute-force algorithm generally requires a lot of processing time. Unfortunately for specific problems, there are still no algorithms that solve the problem precisely besides brute-force ones.

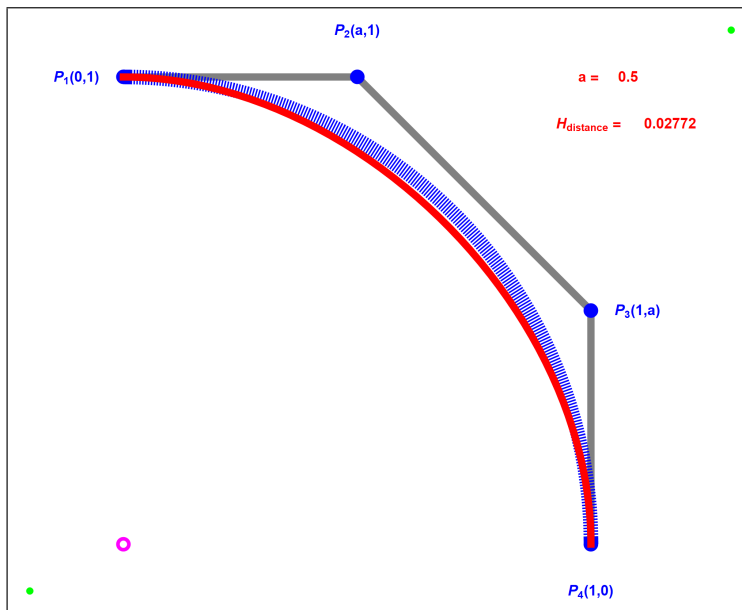
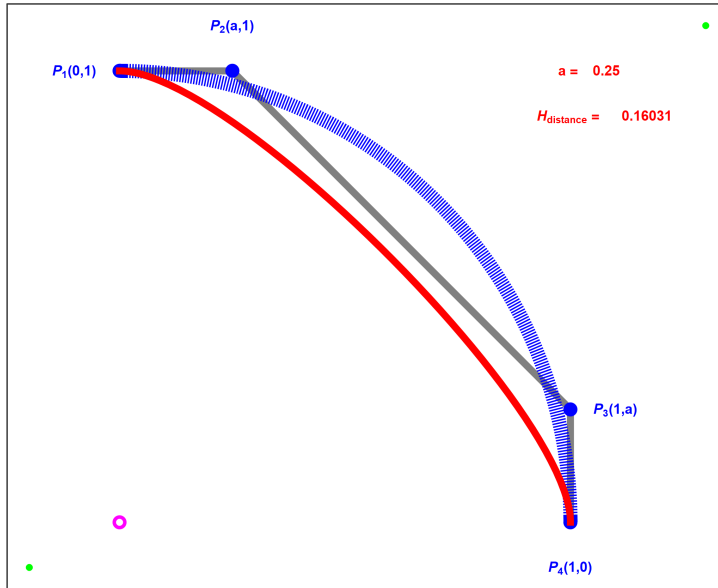
(*MATHEMATICA CODE FOR FINDING THE BEST BEZIER CUBIC*)
(*APPROXIMATION IN THE HAUSDORFF METRIC FOR A CIRCLE*)

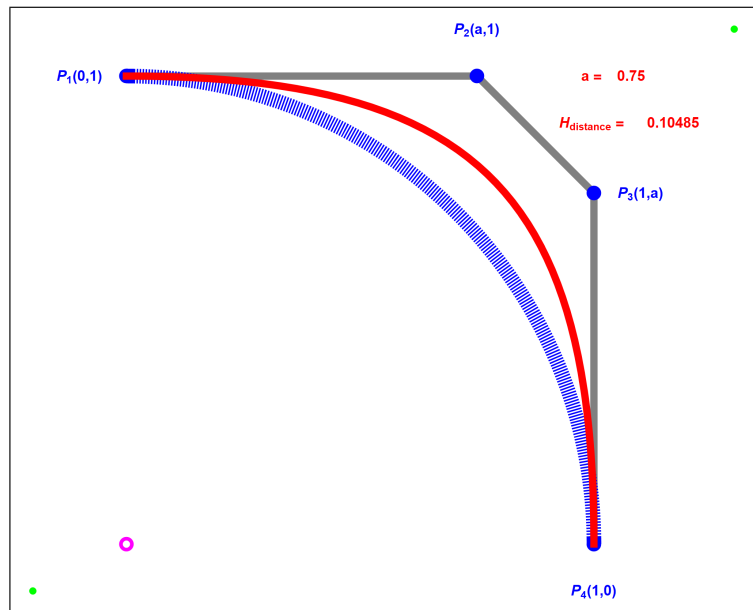
```
(* Define the Bezier points *)
pts[a_]:={{0, 1}, {a, 1}, {1, a}, {1, 0}}
cerc[t_]:= {Sin[tπ/2], Cos[tπ/2]}
Bez[a_, t_]:= (1 - t)3pts[a][[1]] + 3t(1 - t)2pts[a][[2]] + 3t2(1 - t)pts[a][[3]]
+ t3pts[a][[4]]
d[a_, s_, t_]:=√((Bez[a, s][[1]] - cerc[t][[1]])2 + (Bez[a, s][[2]] - cerc[t][[2]])2)
(* Number of the division points *) n = 501;
d1[a_]:=Max [Table [Min [Table [d [a,  $\frac{s-1}{n-1}$ ,  $\frac{t-1}{n-1}$ ], {t, 1, n}]]], {s, 1, n}]]
d2[a_]:=Max [Table [Min [Table [d [a,  $\frac{s-1}{n-1}$ ,  $\frac{t-1}{n-1}$ ], {s, 1, n}]]], {t, 1, n}]]
distH[a_]:=Max[d1[a], d2[a]]
(* Refine the interval of a *) amin = 0.551; amax = 0.553;
distant = N [Table [distH [amin + (amax - amin) $\frac{i-1}{n-1}$ ], {i, 1, n}]]];
minmax = Min[distant];
pozitieCMBA = Position[distant, minmax];
valoareoptimaa = N [amin + (amax - amin) $\frac{\text{pozitieCMBA}[[1]][[1]]-1}{n-1}$ ];
Print [" | aoptim = ", valoareoptimaa, " | Hausdorff distance = |", N[minmax]]

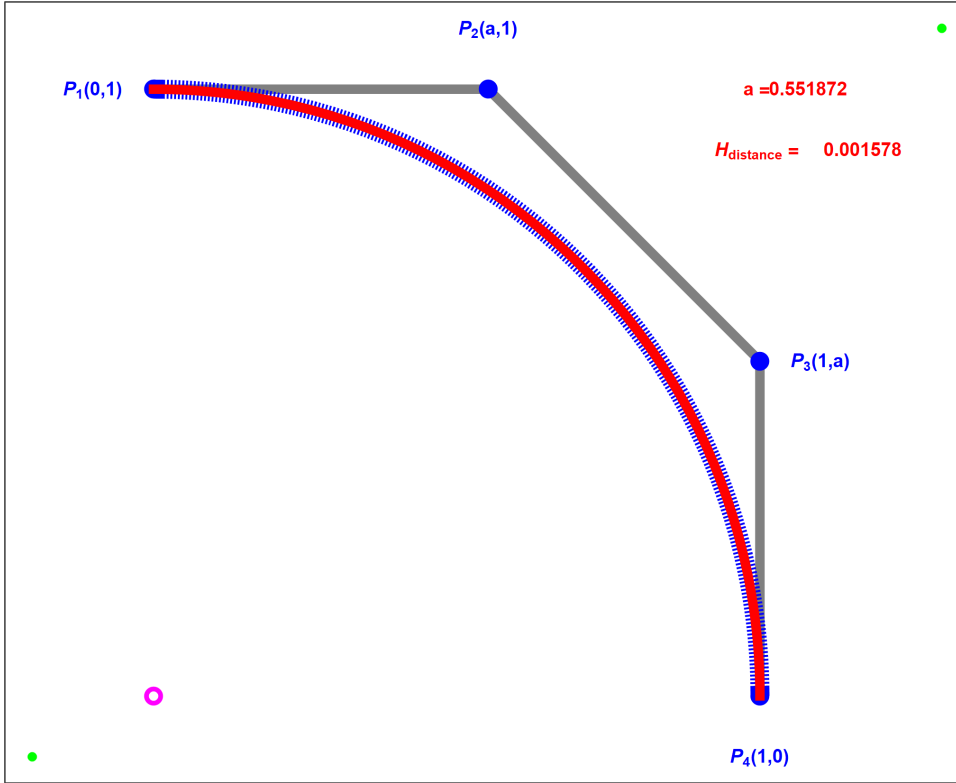
| aoptim = 0.551872 | Hausdorff distance = 0.0015780 |
```

We note that, by replacing the Hausdorff distance with the radial one in our previous MATHEMATICA code, we also checked the value of a obtained by Mortensen.

In the following figures, the cubic Bézier curves are **red-colored**. The circle is traced with a thicker “dashed” line for distinguishing the case when it “overlaps” the Bézier cubic.







Values of the parameter a and the corresponding Hausdorff distances are presented in the following table:

				value of a for $n = 501$
$a =$	0.25	0.5	0.75	0.5518...
$d_H(\mathbf{B}_a, \mathbf{C}) =$	0.1603...	0.0277...	0.1048...	0.001578...

6 The approximation of a plane curve by Bézier cubics

Replacing

$$\mathbf{cerc}[t_]:= \{\mathbf{Sin}[t\pi/2], \mathbf{Cos}[t\pi/2]\}$$

in the previous code by

$$r[t_]:= \{x[t], y[t]\},$$

where r is an arbitrary continuous function, and using an appropriate system of Bézier points, we can obtain estimates of the Hausdorff distance for any continuous curve.

For example, consider the set $\mathbf{COS} = \{(t, \cos(\frac{\pi}{2}t)), t \in [0, 1]\}$. We approximate the set $\mathbf{COS}$ by the Taylor and Hermite polynomials of third degree as well as with the best approximation from a family of the Bézier cubics associated to the system of points

$$(P_1 = (0, 1), P_2 = (a, 1), P_3 = (b, \pi(1 - b)/2), P_4 = (1, 0)),$$

$$a, b \in [0, 1].$$

For $a = 0.3405$, $b = 0.655$, the graph of the Bézier cubic

$$\begin{aligned} \mathbf{Bezier}(a, b, t) = & (t^3 + 1.965(1 - t)t^2 + 1.0215(1 - t)^2t, \\ & 1.62577t^2(1 - t) + (1 - t)^3 + 3t(1 - t)^2), \quad t \in [0, 1], \end{aligned}$$

overlaps the graph of the cos function. We obtain:

$$d_H(\mathbf{Bezier}, \mathbf{COS}) \approx 0.005...$$

Approximation with the Taylor polynomial

$$\mathbf{Taylor}_3 \left[\cos \left(\frac{\pi}{2}t \right); x \right] = 1 - \frac{\pi^2 x^2}{8} + 0 x^3$$

A theoretical bound of the error in the uniform norm in Taylor's formula is:

$$\|\mathbf{Taylor}_3 - \mathbf{COS}\| \leq \frac{\pi^4}{384} = \frac{(\frac{\pi}{2})^4}{4!} = 0.2537...$$

We obtain an estimate of the Hausdorff distance, by comparing a million distances between pairs of points of the sets of the two curves:

$$d_H(\mathbf{Taylor}_3, \mathbf{COS}) \approx 0.2337...$$

Approximation with the interpolation polynomial of Hermite

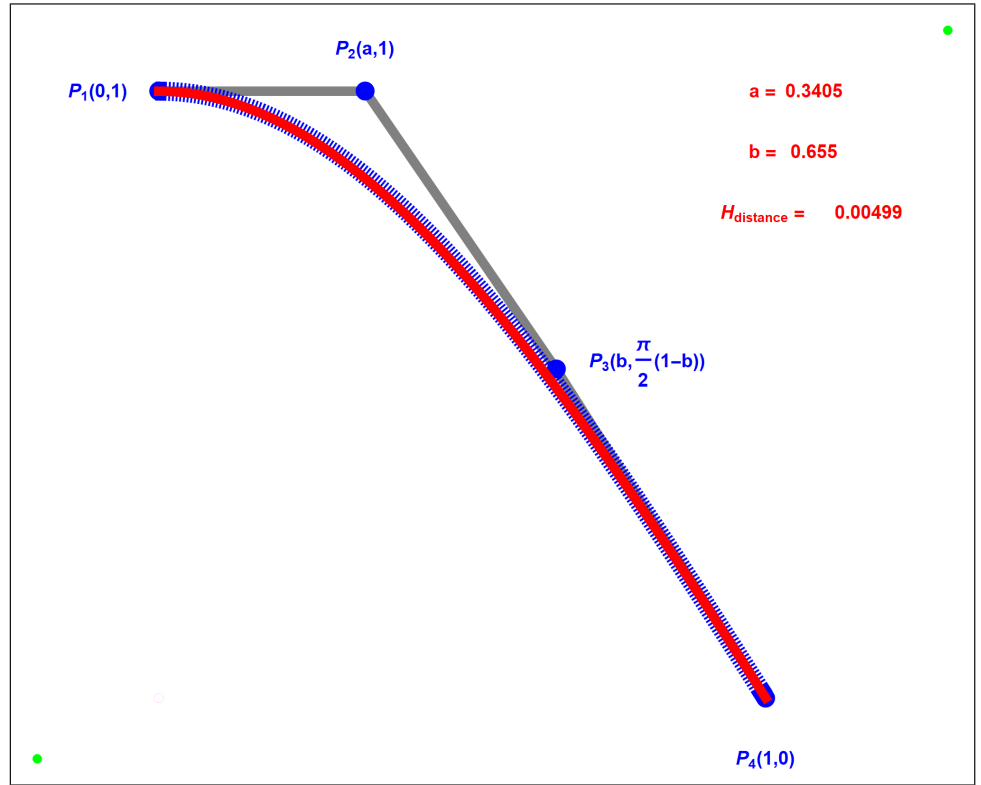
We consider the interpolation polynomial of Hermite on the double knots 0 and 1 for the function $f(t) := \cos\left(\frac{\pi}{2}t\right)$, $t \in [0, 1]$.

$$\begin{aligned} & \mathbf{Hermite}(x) \\ := & \mathbf{Hermite}[f; \{\{0, f[0], f'[0]\}, \{1, f[1], f'[1]\}\}](x) \\ = & \mathbf{Hermite}\left[f; \left\{\{0, 1, 0\}, \left\{1, 0, -\frac{\pi}{2}\right\}\right\}\right](x) \\ = & 1 + \frac{1}{2}x^2(\pi - 6 + 4x - \pi x), \quad x \in [0, 1]. \end{aligned}$$

In summary, we obtained:

$$\begin{aligned} \|\mathbf{Taylor}_3 - \mathbf{COS}\| & \leq 0.2537..., \\ d_H(\mathbf{Taylor}_3, \mathbf{COS}) & \approx 0.2337..., \\ d_H(\mathbf{Hermite}, \mathbf{COS}) & \approx 0.0101..., \\ d_H(\mathbf{Bezier}, \mathbf{COS}) & \approx 0.0049.... \end{aligned}$$

The cubic Bézier curve gives the best approximation.



Approximation of the **cosine** function (dashed line) by a **Bézier cubic**.

Acknowledgements. We thank the reviewer for valuable suggestions and help in revising the manuscript.

References

- [1] Young Joon Ahn. Hausdorff distance between the offset curve of quadratic Bezier curve and its quadratic approximation. *Commun. Korean Math. Soc.*, 22(4):641–648, 2007.
- [2] Young Joon Ahn and Christoph Hoffmann. Circle approximation using LN Bézier curves of even degree and its application. *J. Math. Anal.*

Appl., 410(1):257–266, 2014.

- [3] Young Joon Ahn, Christoph Hoffmann, and Yeon Kim. Curvature continuous offset approximation based on circle approximation using quadratic bezier biarcs. *Computer-Aided Design*, 43:1011–1017, 08 2011.
- [4] Young Joon Ahn, Yeon soo Kim, and Youngsuk Shin. Approximation of circular arcs and offset curves by Bézier curves of high degree. *J. Comput. Appl. Math.*, 167(2):405–416, 2004.
- [5] Samuel Daniel Conte and Carl De Boor. *Elementary numerical analysis: an algorithmic approach*, volume 78. SIAM, 2017.
- [6] Paul de Faget de Casteljau. Outilages méthodes calcul. Technical report, A. Citroen, Paris, 1959.
- [7] Tor Dokken, Morten Dæhlen, Tom Lyche, and Knut Mørken. Good approximation of circles by curvature-continuous Bézier curves. volume 7, pages 33–41. 1990. *Curves and surfaces in CAGD '89* (Oberwolfach, 1989).
- [8] Michael Goldapp. Approximation of circular arcs by cubic polynomials. *Comput. Aided Geom. Design*, 8(3):227–238, 1991.
- [9] Seok Hur and Tae-wan Kim. The best G^1 cubic and G^2 quartic Bézier approximations of circular arcs. *J. Comput. Appl. Math.*, 236(6):1183–1192, 2011.
- [10] Donald E. Knuth. *The METAFONTbook*. Addison Wesley Publishing Company, Reading-Massachusetts, 1986.
- [11] Spencer Mortensen. Approximate a circle with cubic Bézier curves, 2012. <http://spencermortensen.com/articles/bezier-circle/>.
- [12] Tiberiu Popoviciu. *Despre cea mai bună aproximație a funcțiilor continue prin polinoame. Cinci lecții ținute la Facultatea de Științe din Cluj în anul școlar 1933–34 (Romanian). About the best approximation of the continuous functions through polynomials. Five lessons held at the*

Faculty of Science from Cluj in the school year 1933–34 (Romanian).
Inst. Arte Grafice Ardealul, Cluj, 1937.

- [13] Abedallah Rababah. The best uniform quadratic approximation of circular arcs with high accuracy. *Open Math.*, 14(1):118–127, 2016.
- [14] Aleksas Riskus and Giedrius Liutkus. An improved algorithm for the approximation of a cubic Bezier curve and its application for approximating quadratic Bezier curve. *ITC*, 42:303–308, 2013.
- [15] Aleksas Riškus. Approximation of a cubic Bezier curve by circular arcs and vice versa. *Information Technology and Control*, 35, 01 2006.